# ALGORITHMS FOR THE CALCULATION OF HADAMARD-WALSH SPECTRUM FOR COMPLETELY AND INCOMPLETELY SPECIFIED BOOLEAN FUNCTIONS

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*ABSTRACT*

Two new algorithms are described for the calculation of forward Hadamard-Walsh transform for completely and incompletely specified Boolean functions. The first method for the calculation of Hadamard-Walsh transform is based on the direct manipulation on Karnaugh maps. The conversion starts from Karnaugh maps and results in Hadamard-Walsh spectral coefficients. The second algorithm for the calculation of forward transform makes use of the properties of an array of disjoint cubes representation of Boolean functions.

## 1. INTRODUCTION

Spectral techniques in digital logic design have been used for more than thirty years. They have been used for Boolean function classification and the design of logical devices [3-5]. Another area of usage is signal processing, especially image processing and pattern analysis [1]. Spectral methods for testing of logical network by verification of the coefficients in the spectrum have been developed [4].

This paper gives a new insight into spectral methods. By investigating links between spectral techniques and classical logic design methods one are able to present this interesting area of research in a simple manner. The real meaning of spectral coefficients in classical logic terms is shown. Moreover, an algorithm is shown for easily handling the calculation of spectral coefficients for completely and incompletely specified Boolean functions by handwriting manipulations directly from Karnaugh maps. All mathematical relationships between the number of true, false, don't care minterms and spectral coefficients are stated.

One of the drawbacks of spectral techniques is the fact that practically all the existing algorithms for calculating the spectral coefficients start from a form of Boolean function, being either a list of true minterms ( alternatively - a list of false minterms ) or an already minimized sum-of-products Boolean expression. The second new algorithm overcomes this weakness by calculating spectral coefficients for completely and incompletely specified Boolean functions directly from the disjoint cube representation of these functions. Most of the current methods in spectral domain deal only with completely specified Boolean functions. On the other hand, all the algorithms introduced here are valid not only for completely specified Boolean functions but for functions with don't cares as well.

## 2. LINKS BETWEEN SPECTRAL TECHNIQUES AND CLASSICAL LOGIC DESIGN

Let us show more clearly in classical logic terms what is the real meaning of spectral coefficients. Moreover, let us expand our considerations for incompletely specified Boolean functions as well. Since the name of standard trivial function is used in the sequel, the description of such a function follows. Each spectral coefficient $s_I$ gives a *correlation value* between the Boolean function F and a *standard trivial function* $u_I$ corresponding to this coefficient. The standard trivial functions for the spectral coefficients are, respectively, for the dc coefficient $s_I$ ( $I = 0$ ) - the universe of the Boolean function F denoted by $u_0$, for the first order coefficient $s_I$ ( $I = i$ , $i \neq 0$ ) - the variable $x_i$ of the Boolean function F denoted by $u_i$, for the second order coefficient $s_I$ ( $I = ij$, $i \neq 0$, $j \neq 0$ ) - the exclusive-or function between variables $x_i$ and $x_j$ of the Boolean function F denoted by $u_{ij}$, for the third order coefficient $s_I$ ( $I = ijk$, $i \neq 0$, $j \neq 0$, $k \neq 0$ ) - the exclusive-or function between variables $x_i$, $x_j$, and $x_k$ of the Boolean function F denoted by $u_{ijk}$, etc. of Boolean function F, where both the function F and the standard trivial function $u_I$ have the logical values 1;

The following symbols will be used. Let $a$ be the number of true minterms of Boolean function F, where both the function F and the standard trivial function have the logical values 1; let $b_I$ be the number of false minterms of Boolean function F, where the function F has the logical value 0 and the standard trivial function $u_I$ has the logical value 1; let $c_I$ be the number of true minterms of Boolean function F, where the function F has the logical value 1 and the standard trivial function $u_I$ has the logical value 0; let $d_I$ be the number of false minterms of Boolean function F, where both the function F and the standard trivial function $u_I$ have the logical values 0, and $e$ be the number of don't care minterms of Boolean function F.

The value of e is equal to 2 for all Karnaugh maps

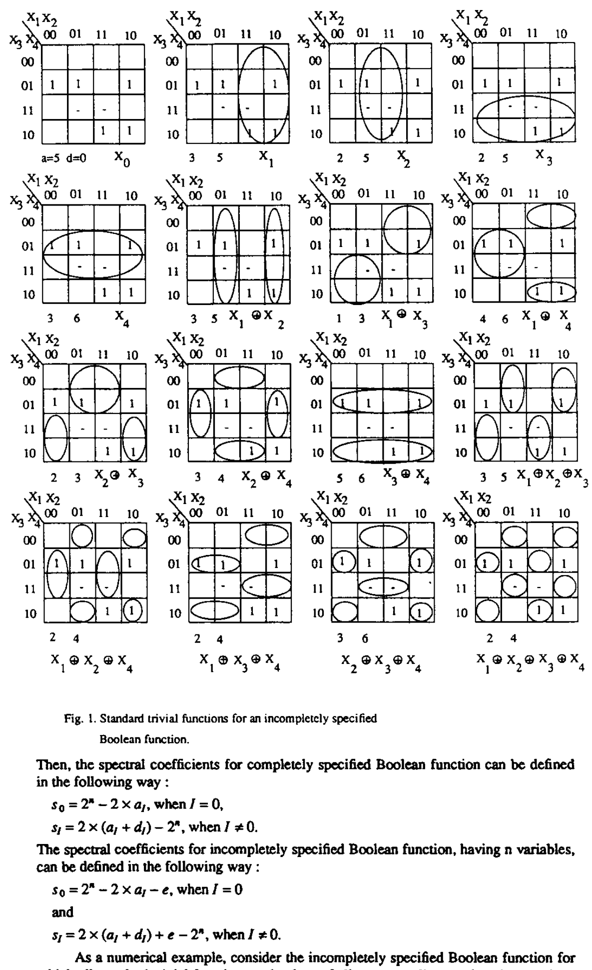


Fig. 1. Standard trivial functions for an incompletely specified Boolean function.

Then, the spectral coefficients for completely specified Boolean function can be defined in the following way :

$s_0 = 2^n - 2 \times a_I$, when $I = 0$,

$s_I = 2 \times (a_I + d_I) - 2^n$, when $I \neq 0$.

The spectral coefficients for incompletely specified Boolean function, having n variables, can be defined in the following way :

$s_0 = 2^n - 2 \times a_I - e$, when $I = 0$

and

$s_I = 2 \times (a_I + d_I) + e - 2^n$, when $I \neq 0$.

As a numerical example, consider the incompletely specified Boolean function for which all standard trivial functions and values of all corresponding $a_I$, $d_I$ and $e$ are given in Fig.1. Then, according to the above formulas, the spectrum of this function is as follows:

$s_0 = 4$ , $s_1 = 2$ , $s_2 = 0$ , $s_3 = 0$ ,
$s_4 = 4$ , $s_{12} = 2$ , $s_{13} = -6$ , $s_{14} = 6$ ,
$s_{23} = -4$ , $s_{24} = 0$ , $s_{34} = 8$ , $s_{123} = 2$ ,
$s_{124} = -2$ , $s_{134} = -2$ , $s_{234} = 4$ , $s_{1234} = -2$ .

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## 3. APPLICATION OF AN ARRAY METHOD TO SPECTRAL COEFFICIENTS CALCULATION FOR COMPLETELY AND INCOMPLETELY SPECIFIED BOOLEAN FUNCTIONS

There exists an algorithm of calculating spectral coefficients for completely specified Boolean functions directly from sum-of-products Boolean expression [4]. In the case, when the implicants are not mutually disjoint, this algorithm requires additional correction for minterms of Boolean function F that are included more than once in some implicants in order to calculate the exact values of spectral coefficients. By using the representation of Boolean functions in the form of an array of disjoint cubes one can apply the existing algorithm without necessity of performing additional correction operations, since for an array of disjoint cubes as an input data to this algorithm the exact spectral coefficients can be calculated immediately. Moreover, we propose the extension of the algorithm for incompletely specified Boolean functions.

In the sequel, the properties of the existing algorithm are rewritten into the notation corresponding to our representation of Boolean functions in the form of arrays of disjoint cubes. All properties describing incompletely specified Boolean functions have never been published.

**Definition.**

A cube of degree m is a cube which has m defined variables that can be either positive or negative (i.e., m is equal to the sum of number of zeroes and ones in the description of a cube).

Suppose we are given arrays of disjoint ON- and DC- cubes that fully define Boolean function F. Then, each cube of degree m can be treated as a minterm within its particular reduced m-space of function F. Let us recall, that the spectrum of each true minterm is given by $s_0 = 2^n - 2$, and all remaining $2^n - 1$ coefficients are equal to $\pm 2$ [4].

The cubes of degree m have the following properties:

1. The contribution of the ON- cube of degree m to full n-space spectrum of function F ( where n is a number of variables in the function F ) is related as follows:

   $s_0$ in full n-space $= 2^n - 2 \times ( 2^{n-m} )$

   and

   $s_I$ in full n-space $= s_I$ in m-space $\times ( 2^{n-m} )$, where $I \neq 0$.

2. The contribution of the DC- cube of degree m to full n-space spectrum of function F is related as follows:

   $s_0$ in full n-space $= 2^{n-1} - 2^{n-m}$

   and

   $s_I$ in full n-space $= s_I$ in m-space $\times 2^{n-m-2}$, where $I \neq 0$.

As one can notice, the contribution of the DC- cube of degree m is equal to one half of the contribution of the ON- cube that has the same degree m. Moreover, the contribution of the ON- or DC- cube of degree m to full n-space spectrum of function F can be expressed for $s_0$ as the absolute value of the sum of all spectral coefficients corresponding to cube of degree m that are negative.

The following properties of the signs of each spectral coefficient $s_I$, where $I \neq 0$ are valid for ON- and DC- cubes of any degree:

1. If in a given cube the $x_I$ variable of Boolean function is in affirmation, then the sign of the corresponding first-order coefficient is positive, otherwise for variable that is in negation, the sign of the corresponding first-order coefficient is negative.
2. The signs of all even-order coefficients are given by the negation of the multiplication of the signs of the related first-order coefficients.
3. The signs of all odd-order coefficients are given by the multiplication of the signs of the related first-order coefficients.

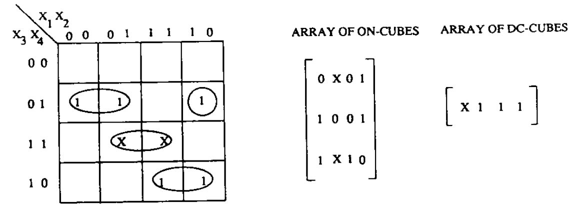
The algorithm is as follows:

**Algorithm.**

**Spectral coefficients calculation for completely and incompletely specified Boolean functions.**

1. For each ON- and DC- cube of degree m calculate the value and the sign of the contribution of this cube to full n-space spectrum according to the properties described previously.
2. The values of all $s_I$ but $s_0$ spectral coefficients are equal to the sum of all contributions to these spectral coefficients from all ON- and DC- disjoint cubes from an array of cubes.
3. The value of DC spectral coefficient $s_0$ is equal for a completely specified Boolean function to the sum of all the corresponding contributions from all ON- disjoint cubes, but it requires the correction factor $-(k-1) \times 2^n$, where $k$ is a number of disjoint cubes in the ON-array of cubes.
4. The value of DC spectral coefficient $s_0$ is equal for an incompletely specified Boolean function to the sum of all the corresponding contributions from all ON- and DC- disjoint cubes, but it requires the correction factor $-(k-1) \times 2^n - l \times 2^{n-1}$, where $k$ is the number of disjoint ON- cubes, and $l$ is the number of disjoint DC- cubes.

Of course, the algorithm can calculate each coefficient separately or in parallel. Should the full of $2^n$ spectral coefficients not be wanted for a particular application, then a reduced set of operations is to be performed.




| CUBE | $S_0$ | $S_1$ | $S_2$ | $S_3$ | $S_4$ | $S_{12}$ | $S_{13}$ | $S_{14}$ | $S_{23}$ | $S_{24}$ | $S_{34}$ | $S_{123}$ | $S_{124}$ | $S_{134}$ | $S_{234}$ | $S_{1234}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 X0 1 | 12 | -4 | | -4 | 4 | | -4 | 4 | | | 4 | | | 4 | | |
| 1 0 0 1 | 14 | 2 | -2 | -2 | 2 | 2 | 2 | -2 | -2 | 2 | 2 | 2 | -2 | -2 | 2 | -2 |
| 1 X 1 0 | 12 | 4 | | 4 | -4 | | -4 | 4 | | | 4 | | | -4 | | |
| X 1 1 1 | 6 | | 2 | 2 | 2 | | | | -2 | -2 | -2 | | | | 2 | |
| SUM | 4 | 2 | 0 | 0 | 4 | 2 | -6 | 6 | -4 | 0 | 8 | 2 | -2 | -2 | 4 | -2 |

$$S_0 = 12 + 14 + 12 + 6 - (3 - 1) * 2^4 - 2^{4-1} = 44 - 32 - 8 = 4$$

Fig. 2. Spectrum S of an incompletely specified Boolean function.

As a numerical example, consider the same incompletely specified Boolean function for which a graphical method was used previously to calculate spectral coefficients. The resulting spectrum is shown in Fig. 2 and as it can be easily checked is exactly the same as the one obtained by graphical method.

## 4. CONCLUSION

New, efficient algorithms for the generation of spectral coefficients have been shown. There is no doubt that the graphical method for calculation of spectral coefficients directly from Karnaugh map is a powerful and efficient tool for functions with variables less than or equal to six. The representation of completely and incompletely specified Boolean function as the arrays of disjoint ON- and DC-cubes is the basis for definitions of the second algorithm for the calculation of Hadamard-Walsh spectrum. Due to properties of Walsh transforms [2,4] the shown methods generate spectral coefficients for any Walsh type of transform. Let us note that by using the proposed methods each coefficient can be calculated separately. Therefore, the methods are very efficient when one wants to calculate only few selected spectral coefficients [4]. This feature of the second algorithm permits also on development of many ways of parallelization of the algorithm which can be implemented on parallel computers or in distributed environments. Moreover, the inherent matrix properties of the algorithm permit for efficient implementation on pipelined vector processors and recently introduced DSP coprocessors for personal computers. It makes it also very well suited for systolic VLSI realizations.


## 5. ACKNOWLEDGMENT

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## 6. LITERATURE


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