1.1 Mathematical description

Uniform transm. lines: dimensions and el. properties are identical at all planes transverse to the direction of propagation.

Length of the transm. line **need not** be negligible compared to the wavelength. Extension of a.c. circuit theory to DISTRIBUTED circuits.

Define resistance, conductance, inductance and capacitance per unit length: R', G', L', C' for uniform transm. line.

Explain difference between R' and G'

- R' related to dimensions and conductivity of (metallic) conductors
- G' related to loss tangent of the insulating material between
- L' related to magnetic flux linking the conductors
- C' related to charge on conductors

Fig. 3-1 (analogy to twin lead or coax)

Δz represents a "small" section of the transm. line — small is relative to the wavelength

Kirchoff's laws for V and I for a small segment (sketch)

$$V = (R^{'}\Delta z)I + (L^{'}\Delta z)\frac{\partial I}{\partial t} + (V + \Delta V) \tag{1}$$

$$I = (I + \Delta I) + (V + \Delta V)(G^{'}\Delta z) + (C^{'}\Delta z)\frac{\partial (V + \Delta V)}{\partial t} \tag{2}$$

division by Δz gives:

$$\frac{V - (V + \Delta V)}{\Delta z} = R^{'}I + L^{'}\frac{\partial I}{\partial t} \tag{3}$$

$$\frac{I - (I + \Delta I)}{\Delta z} = (V + \Delta V)G^{'} + C^{'}\frac{\partial (V + \Delta V)}{\partial t} \tag{4}$$

Taking the limit $\Delta z \to 0$ gives:

$$-\frac{\partial V}{\partial z} = R^{'} I + L^{'}\frac{\partial I}{\partial t} \text{ and } -\frac{\partial I}{\partial z} = V G^{'} + C^{'}\frac{\partial V}{\partial t} \tag{5}$$

Analogous to the eqs. we had for E and H. Combining produces:

$$\frac{\partial^2 V}{\partial z^2} = L^{'} C^{'}\frac{\partial^2 V}{\partial t^2} + (R^{'} C^{'} + G^{'} L^{'})\frac{\partial V}{\partial t} + R^{'} G^{'} V \tag{6}$$

and similar for I.

For perfect conductor $R^{'} = 0$, perfect insulator $G^{'} = 0$

$$\frac{\partial^2 V}{\partial z^2} = L^{'}C^{'}\frac{\partial^2 V}{\partial t^2} \text{ and } \frac{\partial^2 I}{\partial z^2} = L^{'}C^{'}\frac{\partial^2 I}{\partial t^2} \tag{7}$$

These are **wave eqs** and their solutions are "EM wave"-like , traveling along both directions. Both current and voltage waves.

In general solutions will satisfy $f(t \pm \sqrt{L'C'}z)$.

Look at $f(z = 0) = f(t) = V$ and at another $z = z_1$ where $V = f(t - \sqrt{L'C'}z_1)$. By the time $t = \sqrt{L'C'}z_1$ the value inside () is the same as for $z = 0 \Rightarrow$ only difference is time delay of $\sqrt{L'C'}z_1$.

The velocity of propagation of such wave is

$$v = \frac{z_1}{t_d} = \frac{z_1}{z_1\sqrt{L'C'}} = \frac{1}{\sqrt{L'C'}} \tag{8}$$

Note that $f(t + \sqrt{L'C'}z)$ travels in negative direction.

I and V travel with the same velocity.

for EM waves we found that E/H was constant: same here, but instead of η we have

$$\text{CHARACTERISTIC IMPEDANCE } Z_0 = \frac{V}{I} = \sqrt{\frac{L'}{C'}} \tag{9}$$

1.2 Transient behavior

Charging of an infinite transmission line. (fig. 3-2)

Idea of sequential current increases, capacitance charging, current increase etc. Instantaneous changes impossible due to L and C.

V and I generate each other and travel with the same velocity v.

Line infinite $\Rightarrow$ only $+$ directed wave. Since V/I is constant, only $Z_0 (= 100\Omega)$ is seen at all times. $\Rightarrow$ infinite line presents Z_0 load to the battery and it keeps drawing current.

Note that the steady state at the input is reach immediately.

Finite length: open circuit. Fig. 3-3.

At $t = 0, V = 10V$ appears at input of transm. line, i.e. line behaves like and **infinite** line $\Rightarrow$ input impedance $= Z_0$ at $t = 0$.

Input current $I_{in} = V_G/(R_G + Z_0) = 20/200 = 0.1A$ and input voltage $V_{in} = 10V$. These stay constant until later (when reflected wave arrives).

For propagation velocity $v = 2 \cdot 10^8 m/s$, time it takes the wave to get to the end is $t = l/v = 4/2 \cdot 10^8 = 20ns$.

Fig. 3-3 a) shows the situation at half the time it takes to travel to the end, and 3-3 b) is for time just before reaching the end.

Something must happen at the end of the line:

- constant ratio $V^+/I^+ = Z_0$ must be preserved for the traveling wave
- $Z = \infty$ since the end is open-circuited.

Reflection of traveling waves at O-C makes it possible to satisfy both requirements.

For voltage and current at the load we have $V_L = V^+ + V^-$ and $I_L = I^+ - I^-$. Since $I_L = 0 \Rightarrow I^+ = I^-$ and $V^- = I^- Z_0 = I^+ Z_0 = V^+$. Finally, $V_L = 2V^+ =$ 20V.

Define reflection coefficient $\Gamma = V^-/V^+ = I^-/I^+$ and is =1 for O-C case.

Current becomes zero as I^- wave returns and V goes to 20V.

At the generator end $V^-/I^- = Z_0 = 100\Omega$ is the same as the impedance of the generator, so no need for additional requirements to be satisfied $\Rightarrow$ no more reflections.

Resistive termination

Fig. 3-4. part a) is as before. At the end of the line $Z_0 \neq R_L$ and Ohm's law is not satisfied if we also want to keep the Z_0 constant. Solution: reflections must occur at load end.

$V_L = V^+ + V^-$, $I_L = I^+ - I^- = (V^+ - V^-)/Z_0$. Ohm's law $V_L = I_L R_L \Rightarrow$

$$R_L = \frac{V^+ + V^-}{V^+ - V^-} Z_0 = \frac{1+\Gamma_L}{1-\Gamma_L} Z_0 \text{ or } \Gamma_L = \frac{R_L - Z_0}{R_L + Z_0} \qquad (10)$$

In steady-state transm. line is a short circuit.

Analogy with Γ for TEM waves ($= (\eta - \eta_0)/(\eta + \eta_0)$.

For resistance Γ is real but it is complex for other terminations.

$\Gamma = 1$ for O-C, and $\Gamma = -1$ for S-C (short circuit).

For $R_L = Z_0$ no reflections occur since Ohms law is automatically satisfied.

Multiple reflections

Both R_L and $R_G \neq Z_0$! see Fig. 3-5.

At t=0 generator sees Z_0 (as before) $\Rightarrow I^+ = 90/300 = 0.3A$ and $V^+ = 30V$. R_L has no influence.

After 20ns there is a reflection on load end: $\Gamma_L = (25-100)/125 = -0.6$ resulting in $V^- = V^+\Gamma = -18V$.

Total voltage $V_{total} = V^+ + V^- = 12V$. Similarly for reflected and total currents we have $I^- = \Gamma I^+ = -0.18A$ and $I_{tot} = I_L = 0.3 + 0.18 = 0.48A$

These two combined waves reach source end, and there is another reflection ($R_G \neq Z_0$): $\Gamma_G = (R_G - Z_0)/(R_G + Z_0) = V_{refl}/V_{inc} = I_{refl}/I_{inc} = 1/3$.

To find the reflected part $V_{ref} = \Gamma_G V^- = -6V$; etc.

Now the incident (incoming) wave is $V^- = -18V$ which gives $V_{ref} = -6V$ and new total voltage $V_{tot} = 12 - 6 = 6V$

Another reflection at the load end, this time the incident wave is $-6V$ so that $V_{ref} = -6 * (-0.6) = 3.6V$ for the new $V_{tot} = 6 + 3.6 = 9.6V$.

Amplitudes of reflected waves get smaller with each reflection (why?).

Final d.c. value $V = 90 * R_L/(R_G + R_L) = 10V$. Fig. 3-6 shows the time evolution of the input voltage.

Bewley's diagram: voltage and current as functions of time and position. Fig. 3-7. Construct constant z diagram or find value of V or I for fixed t.

For fixed z (=2m) intersections represent arrivals of wave fronts; voltages summed up; forward current waves added but reflected ones subtracted.

Fixed t: sum of voltages at any z above the line gives V variation with z (same for I)

Ringing occurs: period T=80ns (2 x round trip) or $f = 1/T = 12.5$MHz. For $v = 2 \cdot 10^8 \Rightarrow \lambda f = v$ and $\lambda = 16m$ or $l = \lambda/4$.

Interesting result: d.c. source produces high frequency oscillations. If both Γ-s were unity $\Rightarrow$ continuous oscillations, i.e. resonant circuit.

1.3 Sinusoidal excitation

Equations become (using phasor notation):

$$-\frac{d\hat{V}}{dz} = (R^{'} + j\omega L^{'})\hat{I} = Z^{'}\hat{I} \tag{11}$$

$$-\frac{dI}{dz} = (G^{'} + j\omega C^{'})\hat{V} = Y^{'}\hat{V} \tag{12}$$

As for EM waves these are combined:

$$\frac{d^2\hat{V}}{dz^2} = Z^{'}Y^{'}\hat{V} \Rightarrow \hat{V} = \hat{V}_0^{+}e^{-\gamma z} + \hat{V}_0^{-}e^{\gamma z} = V^{+} + V^{-} \tag{13}$$

As before, γ is (complex) propagation constant.

$$\gamma = \sqrt{Z'Y'} = \sqrt{(R' + j\omega L')(G' + j\omega C')} = \alpha + j\beta \tag{14}$$

For current we have:

$$\hat{I} = \hat{I}_0^+ e^{-\gamma z} - \hat{I}_0^- e^{+\gamma z} = \hat{I}^+ - \hat{I}^- \tag{15}$$

Characteristic impedance (voltage to current ratio fro traveling waves)

$$Z_0 = \frac{V_0^+}{I_0^+} = \frac{V_0^-}{I_0^-} = \sqrt{\frac{Z'}{Y'}} = \sqrt{\frac{R' + j\omega L'}{G' + j\omega C'}} \tag{16}$$

Voltages and currents are phasor quantities and depend on conditions on load and generator end.

Power flow : $P = Re(\hat{V}\hat{I}^*) = IV \cos \Theta_{pf}$

Example 3-1.

Low-loss lines

At high frequency: $R' << \omega L'$ and $G' \ll \omega C'$. Almost always true in MW range.

$$Z_0 = \sqrt{\frac{L'}{C'}}\Omega, \quad \alpha \approx \frac{R'}{2Z_0} + \frac{G' Z_0}{2} \text{ Np/length} \tag{17}$$

$$\beta \approx \omega\sqrt{L'C'} \quad v = \frac{1}{\sqrt{L'C'}} \quad \lambda = \frac{\lambda_0}{c\sqrt{L'C'}} \tag{18}$$

Conclusions:

- Eqs. for $Z_0, v, \beta, \lambda \approx$ the same as for lossless lines.

- Since Z_0 is practically real, the average power flow in a **traveling** wave at any point z is product of rms voltage and current $P^+ = V^+I^+$ and $P^- = V^-I^-$.
- For single frequency signals $\Rightarrow$ finite $R^{'}, G^{'}$ introduce attenuation on propagating waves ($e^{-\alpha \Delta z}$ for power it's square).

Similar to EM waves in lossy dielectric. Reason: there is a wave associated with sinusoidal waves but now it is **guided**.

Coaxial transmission lines

Fig. 3-8 shows E and H field distribution.

$$E = \frac{V}{r \ln \frac{b}{a}} \quad H = \frac{I}{2\pi r} \tag{19}$$

Average power flow:

$$P = \int_S \vec{\mathcal{E}} \times \vec{\mathcal{H}} \cdot d\vec{s} \tag{20}$$

In cylindrical coordinates: $d\vec{s} = r \; d\phi \; dr$ ($d\vec{s}$ is element normal to the plane of paper, i.e. cross-sectional area; sketch)

$$P = \int_a^b \int_0^{2\pi} \frac{VI}{2\pi r^2} \left(\ln \frac{b}{a} \right)^{-1} \; r \; d\phi dr = VI \tag{21}$$

(why these limits?)

Interpretation: power flow associated with the voltage and current waves is just another way of viewing EM propagation in the region between conductors!

From early on: $C = 2\pi\epsilon_0\epsilon_r l/\ln(b/a)$. Divide by l to get C'.

Similarly $L = \mu_0\mu_r l \ln(b/a)/(2\pi)$.

$$v = \frac{1}{\sqrt{L'C'}} = \frac{1}{\sqrt{\mu_0\mu_r\epsilon_0\epsilon_r}} = \frac{c}{\sqrt{\mu_r\epsilon_r}} \text{ (familiar?)} \tag{22}$$

$$Z_0 = \sqrt{\frac{L'}{C'}} = \sqrt{\frac{\mu_0\mu_r}{4\pi^2\epsilon_r\epsilon_0}} \ln\frac{b}{a} = 60\sqrt{\frac{\mu_r}{\epsilon_r}} \ln\frac{b}{a} \tag{23}$$

1.4 Terminated transm. lines

Discussion of various loads on transm. lines.

Alternative view: tr. line terminated in its Z_0 has no reflections — based on infinite line picture. Conclusion: line terminated in its Z_0 is equivalent to infinitely long line and has no reflections. Input impedance $= Z_0$.

Fig. 3–9, $\hat{V}_{in}, \hat{I}_{in}$ at point z=0:

$$\hat{V}_{in} = \frac{Z_0}{Z_G + Z_0}\hat{V}_G \quad \hat{I} = \frac{\hat{V}_G}{Z_G + Z_0} \tag{24}$$

No reflections present $\Rightarrow \hat{V}_{in} = \hat{V}_0^+, \hat{I}_{in} = \hat{I}_0^+$.

$\hat{V}, \hat{I}$ still vary along the line:

$$\hat{V} = \hat{V}_0^+ e^{-\gamma z} \quad \hat{I} = \hat{I}_0^+ e^{-\gamma z} \tag{25}$$

At the load end this gives (z=l):

$$\hat{V}_L = \frac{Z_0}{Z_G + Z_0} \hat{V}_G e^{-\alpha l} \angle - \beta l \quad \hat{I}_L = \frac{\hat{V}_G}{Z_G + Z_0} e^{-\alpha l} \angle -\beta l \tag{26}$$

What's the power absorbed by load? $P_L = V_L I_L$ (rms values) since Z_0 is real.

$$P_L = Z_0 \left| \frac{V_G}{Z_G + Z_0} \right|^2 e^{-2\alpha l} \tag{27}$$

magnitude needed since Z_G may be complex

General case (with reflections)

If $Z_L \neq Z_0 \Rightarrow$ reflections. Starting point is

$$\hat{V} = \hat{V}_0^+ e^{-\gamma z} + \hat{V}_0^- e^{+\gamma z} = \hat{V}^+ + \hat{V}^- \quad \hat{I} = \hat{I}_0^+ e^{-\gamma z} - \hat{I}_0^- e^{+\gamma z} \tag{28}$$

Reflection coeff. Γ was defined only at the load or generator end — no need for such a restriction $\Rightarrow$ define it anywhere on line:

$$\Gamma(z) = \frac{\text{refl. V or I}}{\text{incident V or I}} = \frac{\hat{V}^-}{\hat{V}^+} = \frac{\hat{I}^-}{\hat{I}^+} \tag{29}$$

At load end we have:

$$\Gamma_L = \frac{\hat{V}_0^- e^{\gamma l}}{\hat{V}_0^+ e^{-\gamma l}} = \frac{\hat{V}_0^-}{\hat{V}_0^+} e^{2\gamma l} \tag{30}$$

For convenience: transform coordinated so that zero is at load end and + direction is **towards** generator (Fig. 3–10). New coordinate: d=l-z .

Voltage wave somewhere on the tr. line:

$$\hat{V} = V_0^+ e^{-\gamma l} \left[\underbrace{e^{\gamma d}}_{\text{gen.}\to\text{load}} + \underbrace{\Gamma_L e^{-\gamma d}}_{\text{load}\to\text{gen.}} \right] \tag{31}$$

$$\hat{I} = \hat{I}_0^+ e^{-\gamma l} \left[e^{\gamma l} - \Gamma_L e^{-\gamma d} \right] \tag{32}$$

Γ at any point is:

$$\Gamma = \Gamma_L e^{-2\gamma d} = \Gamma_L e^{-2\alpha d} \angle -2\beta d = |\Gamma_L|\, e^{-2\alpha d} \angle (\phi_L - 2\beta d) \quad (33)$$

Interpretation: Input wave attenuated by $e^{-\alpha d}$ on the way to load; phase shifted by $-\beta l$. After reflection attenuated by another $e^{\alpha d}$ $\Rightarrow$ total=$e^{-2\alpha d}$. Only a fraction reflected, as determined by Γ_L.

On input terminals the mag. of refl. wave is $|\Gamma_L| e^{-2\alpha l}$. Phase shift $= -2\beta l$, but Γ_L adds its own $\Rightarrow$ total $= \phi_L - 2\beta l$.

For $\alpha = 0$, $|\Gamma|$ is independent of d, but phase angle is still function of d.

$$d = 0 \Rightarrow \hat{V}_L = \hat{V}_0^+ e^{-\gamma l}\left[1 + \Gamma_L\right] \quad \hat{I}_L = \hat{I}_0^+ e^{-\gamma l}\left[1 - \Gamma_L\right] \tag{34}$$

$$\Rightarrow Z_L = \frac{\hat{V}_L}{\hat{I}_L} = Z_0 \frac{1 + \Gamma_L}{1 - \Gamma_L} \text{ or } \frac{Z_L}{Z_0} = \bar{Z}_L = \frac{1 + \Gamma_L}{1 - \Gamma_L} \tag{35}$$

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{\bar{Z}_L - 1}{\bar{Z}_L + 1} \text{ also } \Gamma_L = \frac{1 - \bar{Y}_L}{1 + \bar{Y}_L} \tag{36}$$

where $Y_L = 1/Z_L$

Analogous to equations in transient case. Difference is in steady state a.c. analysis $\Rightarrow$ valid for any complex load. Also, merger of a.c. circuit theory (impedance) and wave theory (refl. coefficient).

Extend the concept to any point on the tr. line:

$$Z = \frac{\hat{V}}{\hat{I}} = Z_0 \frac{1+\Gamma(d)}{1-\Gamma(d)} \quad \text{and} \quad Y = Y_0 \frac{1-\Gamma(d)}{1+\Gamma(d)} \tag{37}$$

Represents total Z or Y to the right of point d.

Reflection means standing waves. When $Z_L \neq Z_0 \Rightarrow \Gamma_L \neq 0$ and standing waves exist.

Take the short circuit (S-C) load: same as wave reflection of perfect conductor. see fig. 2–24, 2–25. Successive minima are $\lambda/2$ apart, I shifted by $\lambda/4$ relative to V.

General case: $\hat{V}$ is sum of phasor quantities $\hat{V}^+ + \hat{V}^-$.

Lossless case: two counter-rotating vectors of fixed magnitude (fig. 2–28). When in phase $\rightarrow$ maximum

$$V_{max} = V^+ + |\Gamma|V^+ = (1 + |\Gamma|)V^+ \tag{38}$$

$\lambda/4$ away there is a minimum (phasors $180°$ out of phase)

$$V_{min} = V^+ - |\Gamma|V^+ = (1 - |\Gamma|)V^+ \tag{39}$$

Standing wave ratio

$$\text{SWR} = \frac{V_{max}}{V_{min}} = \frac{1 + |\Gamma|}{1 - |\Gamma|} \quad \text{and} \quad |\Gamma| = \frac{\text{SWR} - 1}{\text{SWR} + 1} \tag{40}$$

For lossless line SWR=const., but not otherwise (not useful).

example 3.2

Special Cases

A) Short circuit: $Z_L = 0 \Rightarrow \Gamma_L = -1,\ SWR = \infty$. At d=0 (z=l) V=0; other nulls occur at multiple of $\lambda/2$.

I is maximum at the load (d=0), its rms value is $2I_0^+$.

Since V and I are displace by $\lambda/4$, at $d = \lambda/4$, we have $Z \to \infty$, i.e. short circuit is transformed into open circuit !!!

B) Open circuit: $Y_L = 0,\ \Gamma_L = 1,\ SWR = \infty$. See fig. 3–11 for standing waves. At d=0, I=0; $V_{max} = 2V_0^+$ and at $d = \lambda/4$ we have $Z = 0$, i.e. O-C becomes S-C !!!

C) Reactive loads:

$$Z_L = jX \Rightarrow \Gamma_L = \frac{j\bar{X} - 1}{j\bar{X} + 1} = 1\angle\pi - 2\arctan\bar{X} \qquad (41)$$

For $\underbrace{\bar{X} = +1}_{??} \Rightarrow \Gamma_L = 1\angle 90°$; for $\underbrace{\bar{X} = -1}_{??} \Rightarrow \Gamma_L = 1\angle -90°$. In either case $|\Gamma| = 1, SWR \to \infty$.

Interpret: reactance cannot absorb power $\Rightarrow$ all power in incident wave must be reflected (analogous to S-C and O-C in this respect). Consequence: $|\Gamma_L| < 1$ only if Z_L has a resistive component.

Fig. 3–12 for SW pattern ($Z_L = jZ_0$); first voltage null is $3\lambda/8$ away from d=0

D) Resistive loads:

$$\Gamma_L = \frac{R_L - Z_0}{R_L + Z_0} \Rightarrow \text{ real} \tag{42}$$

For $R_L = Z_0 \Rightarrow$ no reflections.

For $R_L > Z_0 \Rightarrow \Gamma_L > 0$, V_{max} and I_{min} are at the load. (why?) SWR $= R_L/Z_0$.

For $R_L < Z_0 \Rightarrow \Gamma_L < 0$, and V_{min}, I_{min} at the load. SWR $= Z_0/R_L$

SWR $= R_L/Z_0$ or Z_0/R_L whichever is > 1.

$|\Gamma_L| < 1$, i.e. some power is absorbed.

1.5 Power flow

Generally $P = \text{Re}(\hat{V}\hat{I}^*) = VI\cos\Theta_{pf}$

Direction of flow from generator to load $\hat{V} = \hat{V}_0^+ e^{-\gamma z} + \hat{V}_0^- e^{\gamma z} = \hat{V}^+(1+\Gamma)$ and $\hat{I} = \hat{I}^+ - \hat{I}^- = I^+(1-\Gamma)$

The power becomes

$$\begin{aligned} P &= \text{Re}\left[(1+\Gamma)(1-\Gamma)^*\hat{V}^+\hat{I}^{+^*}\right] \\ &= \text{Re}\left[(1+\Gamma)(1-\Gamma^*)\frac{\hat{V}^+V^{+^*}}{Z_0^*}\right] \\ &= \text{Re}\left[(1+\Gamma-\Gamma^*-|\Gamma|+^2)\frac{|V^+|^2}{Z_0^*}\right] \end{aligned} \tag{43}$$

Low-loss, high freq. $\Rightarrow Z_0$ is real $\Rightarrow \Gamma - \Gamma^*$ is imaginary.

$$P = (1 - |\Gamma|^2)\frac{|V^+|^2}{Z_0} = (1 - |\Gamma|^2)P^+ = P^+ - |\Gamma|^2 P^+ = P^+ - P^-$$

For Z_0 real net power flow at any point on the line is the difference between the power in the **forward** wave P^+ and in the **reflected** wave P^-. Note: Γ is function of position.

How about input terminals:

$$P_{in} = P_{in}^+(1 - |\Gamma_{in}|^2) \quad P_L = P_L^+(1 - |\Gamma_L|^2) \tag{44}$$

Remember that $+$ and - signs indicate **traveling** waves.

$$P_{in}^+ = (\overbrace{V_0^+}^{magn.})^2/Z_0, \quad P_L^+ = {V_L^+}^2/Z_0 = P_{in}^+ e^{-2\alpha l} \tag{45}$$

i.e. incident power at the load is input power attenuated along the line of length l. For **lossless** lines $\Rightarrow P_{in}^{+} = P_{L}^{+} = P^{+}$

NET power flow is position dependent and we need to determine P^{+} at any point.

For real Z_0, $\hat{V}^{+}, \hat{I}^{+}$ are **in phase** $\Rightarrow P^{+} = \frac{{V^{+}}^{2}}{Z_0} = \frac{{V_0^{+}}^{2} e^{-2\alpha z}}{Z_0}$

$\hat{V}_0^{+}$ and its rms value V_0^{+} are obtained from Kirchoff's laws at input:

$$\hat{V}_G = \hat{V}_{in} + \hat{I}_{in} Z_G = (\hat{V}_0^{+} + \Gamma_{in} \hat{V_0}^{+}) + Z_G \frac{\hat{V}_0^{+} - \Gamma_{in} \hat{V}_0^{+}}{Z_0} \quad (46)$$

$$\hat{V}_0^+ = \frac{\hat{V}_G Z_0}{(Z_G + Z_0)(1 - \Gamma_G \Gamma_L e^{-2\gamma l})} \tag{47}$$

where $\Gamma_L = (Z_G - Z_0)/(Z_G + Z_0)$

Given $\hat{V}_G, Z_G, Z_L$ (circuit) and Z_0, γ, l (line) we can find V_0^+.

Alternative form:

$$\hat{V}_0^+ = \frac{\hat{V}_G(1 - \Gamma_G)}{2(1 - \Gamma_G \Gamma_{in})} = \frac{\hat{V}_G(1 - \Gamma_G)}{2(1 - \Gamma_G \Gamma_L e^{-2\gamma l})} \tag{48}$$

With V_0^+ known P can be determined.

Two different approaches: a) multiple reflection, b) steady state a.c. circuit. What's the relationship?

case 1) Either $\Gamma_L = 0$ or $\Gamma_G = 0$ so no multiple reflections occur. Initial forward traveling wave from voltage divider $Z_G + Z_0$, $\Rightarrow \hat{V}_0^+ = \hat{V}_G Z_0/(Z_G + Z_0)$. Same as in eq. (47).

case 2) Multiple reflections. $\hat{V}_0^+$ represents phasor sum of all the forward traveling waves at input (z=0).

Initial wave: $\hat{V}_0^+(1) = \hat{V}_G Z_0/(Z_G + Z_0)$

2nd one: $\hat{V}_0^+(2) = \Gamma_G \Gamma_L e^{-2\gamma l} \hat{V}_G Z_0/(Z_G + Z_0)$

3rd one: $\hat{V}_0^+(3) = (\Gamma_G \Gamma_L e^{-2\gamma l})^2 \hat{V}_G Z_0/(Z_G + Z_0)$

etc.

Final result is the sum of all of the above; limit of the series is (when $|\Gamma_L \Gamma_G e^{-2\gamma l}| < 1$) is $(1 - \Gamma_L \Gamma_G e^{-2\gamma l})^{-1}$. This leads to the same expression as in a.c. theory (eq. (47)).

Back to power calculation: $P^+ = {V_0^+}^2 e^{-2\alpha z}/Z_0$. Now plug in expr. for V_0^+ (eq. 48))

$$P^+ = \frac{V_G^2}{4Z_0} \left| \frac{1 - \Gamma_G}{1 - \Gamma_G \Gamma_{in}} \right|^2 e^{-2\alpha z} \tag{49}$$

so, for z=0 we have

$$P_{in}^+ = \frac{V_G^2}{4Z_0} \left| \frac{1 - \Gamma_G}{1 - \Gamma_G \underbrace{\Gamma_{in}}_{=\Gamma_L e^{-2\gamma l}}} \right|^2 \tag{50}$$

Useful concept: available power from source $P_A = V_G^2/(4R_G)$ where $R_G = \text{Re}(Z_G)$

$$P_{in}^{+} = P_A \frac{1 - |\Gamma_G|^2}{|1 - \Gamma_G \Gamma_{in}|^2} \tag{51}$$

Finally, what is the NET input power?

$$P_{in} = P_{in}^{+}(1 - |\Gamma_{in}|^2) = P_A \frac{(1 - |\Gamma_G|^2)(1 - |\Gamma_{in}|^2)}{|1 - \Gamma_G \Gamma_{in}|} \tag{52}$$

Power deliver to the load:

$$P_L = P_A \frac{(1 - |\Gamma_G|^2)(1 - |\Gamma_L|^2)}{|1 - \Gamma_G \Gamma_L e^{-2\gamma l}|^2} e^{-2\alpha l} \tag{53}$$

or for lossless line:

$$P_{in} = P_L = P_A \frac{(1 - |\Gamma_G|^2)(1 - |\Gamma_L|^2)}{|1 - \Gamma_G \Gamma_L e^{-j2\beta l}|^2} \tag{54}$$

In this case $|\Gamma_{in}| = |\Gamma_L|$ then $\Gamma_{in} = \Gamma_L e^{-j2\beta l}$

Assume that the input is matched $Z_G = Z_0$ then $\Gamma_G = 0$ and P_L independent of frequency and phase angle βl ($\beta = \omega / v$).

Therefore, for matched input $\hat{V}_0^+ = \hat{V}_G / 2$ and $P_{in}^+ = P_A = V_G^2/(4Z_0)$ so that

$$P_{in} = P_A(1 - |\Gamma_{in}|^2) \text{ and } P_L = P_A e^{-2\alpha l}(1 - |\Gamma_L|^2) \tag{55}$$

Lossless line, generator end matched; if load is matched as well $\Rightarrow$ all of P_A is delivered to load $P_L = P_A$.

For $Z_G \neq Z_0$ it **does not follow** that matching the load will produce $P_L = P_A$ (see eq. 54). Maximizing power transfer more complicated.

Example 3-3.

Return loss: $L_R = 10\log(P^+/P^-)$ dB. Since $P^- = |\Gamma|^2 P^+$

$$L_R = \log\frac{1}{|\Gamma|^2}\ \text{dB} \tag{56}$$

For lossless line L_R is the same anywhere (i.e. $|\Gamma|$ is independent of position).

Lossy line means position dependent L_R, but it is easily determined from $|\Gamma| = |\Gamma_L| e^{-2\alpha d}$. For input we have $\Gamma_{in} = |\Gamma_L| e^{-2\alpha l} \angle \phi_L - 2\beta l$ so that $L_{R,in} = L_{R,load} + 2(8.686\alpha l)$

<u>Reflection loss:</u> used only when $Z_G = Z_0$. Measures reduction in load power due to mismatch of load impedance ($Z_L \neq Z_0$)

$$\text{Ref. loss} = 10 \log \frac{\overbrace{P_L(Z_L = Z_0)}^{=P_A e^{-2\alpha l}}}{\underbrace{P_L(Z_L)}_{=P_A e^{-2\alpha l}(1-|\Gamma_L|^2)}} \text{ dB} = 10 \log \frac{1}{1 - |\Gamma_L|^2} \quad (57)$$

aka mismatch loss.

1.6 Impedance transformation

Reference direction; impedance anywhere on line replaces everything to the right.

Starting point:

$$Z = Z_0 \frac{1+\Gamma}{1-\Gamma} = Z_0 \frac{1+\Gamma_L e^{-2\gamma d}}{1-\Gamma_L e^{-2\gamma d}} \tag{58}$$

After some manipulation (note: bar in, e.g. $\bar{Z} = Z/Z_0$ is for normalized values):

$$Z = Z_0 \frac{Z_L + Z_0 \tanh \gamma d}{Z_0 + Z_L \tanh \gamma d}, \quad \bar{Z} = \frac{\bar{Z}_L + \tanh \gamma d}{1 + \bar{Z}_L \tanh \gamma d} \tag{59}$$

or for admitance

$$Y = Y_0 \frac{Y_L + Y_0 \tanh \gamma d}{Y_0 + Y_L \tanh \gamma d}, \quad \bar{Y} = \frac{\bar{Y}_L + \tanh \gamma d}{1 + \bar{Y}_L \tanh \gamma d} \tag{60}$$

Input impedance: set d=l; what about d=0?

If α is neglected

$$Z = Z_0 \frac{Z_L + jZ_0 \tan \beta d}{Z_0 + jZ_L \tan \beta d}, \quad \bar{Z} = \frac{\bar{Z}_L + j \tan \beta d}{1 + j\bar{Z}_L \tan \beta d} \tag{61}$$

and equivalent for Y-s

Example 3-4.

Dramatic changes in Z; standing waves are the cause. No SW $\Rightarrow$ no impedance transformation ($Z_L = Z_0 \Rightarrow Z = Z_0$ anywhere).

What happens at LF? $l \ll \lambda \Rightarrow \tan \beta l \approx 0$ so that $Z = Z_L$ for LF and low loss line.

Fixed length $\Rightarrow$ freq. dependence of Z since $\beta l = \omega l / v = 2\pi l / \lambda$

Excellent for impedance transf. but bad for broadband design!

$\underline{\lambda/2 \text{ line:}}$ $l = n\lambda/2$ then $Z_{in} = Z_L$ (substitute $\beta l = n\pi$ into eq. 59).

Impedance repeats itself every $\lambda/2$!!

Take $Z_G = Z_L$, both real (e.g. 20 Ohms). Tr. line has $Z_0 = 100$ Ohms and $l = \lambda/2$. $SWR = Z_0/R = 5$ but for this lenght of tr. line $Z_{in} = Z_L = 20$ Ohms. All available power is delivered down the lossless line to Z_L. Despite SW max. power is delivered to the load.

<u>$\lambda/4$ line:</u> $l = (2n-1)\lambda/4$. Here $\tan \beta l \rightarrow \infty$ and $Z_{in} = Z_0^2/Z_L$ or normalized $\bar{Z}_{in} = 1/\bar{Z}_L$.

Large $Z_L \rightarrow$ small Z_{in} and inverse.

Inductive $Z_L \rightarrow$ capacitive Z_{in} (and inverse)

If Z_L behaves as a series resonant circuit Z_{in} is like parallel res. circuit.

For Z_L real it behaves like a transformer with turns ratio $n_t = 5(= SWR)$ for $Z_L < Z_0$. But for $Z_L > Z_0, n_t < 1$.

Useful for matching a resistive load to generator (to deliver all available power). For Z_G real, Z_{in} must be real and $Z_{in} = R_G$ or $Z_0 = \sqrt{R_G R_L}$.

Since line is assumed lossless all power is delivered to the load.

Problem: works only at one (narrow range of) frequency.

Short circuit: obviously $Z_{in} = jX_{in} = jZ_0 \tan \beta l$. See fig. 3-14

For $\beta l < \pi/2$ (or $l < \lambda/4$) Z_{in} is inductive.

$$L_{eq} = X_{in}/\omega = Z_0 \tan(\beta l)/\omega \approx \underbrace{Z_0 l/v = L' l}_{\text{for } l<0.08\lambda} \tag{62}$$

note that L_{eq} is freq. dependent. Think of this circuit as one turn of a coil (L).

Near $\beta l = n\pi/2$ and odd multiples S-C behaves like parallel resonant circuit.

Near $\beta l = n\pi$ S-C behaves like a series resonant circuit. see fig. 3-15.

Infinite number of resonances. Note on lenght: go for the shortest one ...

Open circuit: use admitances $Y_L = 0$ and $Y_{in} = jB_{in} = jY_0 \tan \beta l$ or $Z_{in} = -jZ_0 \cot \beta l$. Fig. 3-16;

For $l < \lambda/4$

$$C_{eq} = Y_0 \tan(\beta l)/\omega[F] \approx \underbrace{Y_0 l/v}_{l<0.08\lambda} = l/(Z_0 v) = C' l \qquad (63)$$

again it is freq. dependent; it looks like a parallel plate capacitor.

For $\beta l = \pi = 2\pi l/\lambda$ (i.e. $l = \lambda/2$) we have $Y_{in} = 0$ (O-C), but for $l = (2n-1)\lambda/4$, $Y_{in} = \infty$ (S-C).

1) near $\beta l = (2n-1)\pi/2$ O-C behaves like series resonant circuit

2) near $\beta l = n\pi$ O-C looks like parallel resonant circuit.