Just a reminder of various definitions related to transmission lines.

- voltage and current along a transmission line

$$v(x,t) = Re[V(x)e^{j\omega t}] \text{ and } i(x,t) = Re[I(x)e^{j\omega t}] \qquad (1)$$

- characteristic (complex) impedance of the transmission line

$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}} \qquad (2)$$

- uniform trans. lines

$$V(x) = Ae^{-\gamma x} + Be^{\gamma x} \text{ and } I(x) = \frac{A}{Z_0}e^{-\gamma x} - \frac{B}{Z_0}e^{\gamma x} \qquad (3)$$

- propagation constant
  $\gamma = \alpha + j\beta = \sqrt{(R + j\omega L)(G + j\omega C)}$
- phase velocity: $v_p = \omega/\beta$
- incident wave: $e^{-\gamma x} = e^{-\alpha x} e^{-j\beta x}$
- reflected wave: $e^{\gamma x} = e^{\alpha x} e^{j\beta x}$
- electrical length of the line: βx
- reflection coefficient:

$$\Gamma(x) = \frac{B e^{\gamma x}}{A e^{-\gamma x}} = \Gamma_0 e^{2\gamma x} \quad \text{where } \Gamma_0 = \Gamma(x=0) = \frac{B}{A} \quad (4)$$

Input impedance

$$Z_{in} = \frac{V_x}{I_x} = Z_0 \frac{e^{-\gamma x} + \Gamma_0 e^{\gamma x}}{e^{-\gamma x} - \Gamma_0 e^{\gamma x}} \tag{5}$$

At x=0 $\quad Z_{in}(0) = Z_L = Z_0 \frac{1+\Gamma_0}{1-\Gamma_0}$ or $\Gamma_0 \frac{Z_L - Z_0}{Z_L + Z_0}$ (6)

Z_{in} at input terminals of transm. line, x=d

$$Z_{in}(d) = Z_0 \frac{Z_L + Z_0 \tanh(\gamma d)}{Z_0 + Z_L \tanh(\gamma d)} \tag{7}$$

(note the change of positive direction)

Lossless transm. line: $\alpha = 0, \gamma = j\beta$, $\beta = \omega\sqrt{LC}$, $v_p = 1/\sqrt{LC}$, $\lambda = v_p/f$, $Z_0 = \sqrt{L/C}$. Also

$$Z_{in}(d) = Z_0 \frac{Z_L + jZ_0 \tan(\beta d)}{Z_0 + jZ_L \tan(\beta d)} \tag{8}$$

From now on: tr. lines are lossless and uniform (unless specified otherwise).

(Voltage) standing wave ratio:

$$\text{VSWR} = \frac{|V(x)|_{max}}{|V(x)|_{min}} = \frac{1 + |\Gamma_0|}{1 - |\Gamma_0|} \quad \text{or} \quad |\Gamma_0| = \frac{\text{VSWR} - 1}{\text{VSWR} + 1} \tag{9}$$

Matched tr. line: $\Gamma_0 = 0$, $Z_{in} = Z_0$, VSWR=1.

Shorted tr. line (Z_L =0): $\Gamma_0 = -1$, VSWR=∞, $Z_{in}(d) = Z_{sc}(d) = jZ_0 \tan(\beta d)$

Open circuited tr. line: $Z_L = \infty$: $\Gamma_0 = 1$, VSWR=∞, $Z_{in}(d) = Z_{oc}(d) = -jZ_0 \cot(\beta d)$

Quarter wave tr. line (transformer), $d = \lambda/4$, or $\beta d = \pi/2$ (see eq. 8) $\Rightarrow Z_{in}(d) = Z_{in}(\lambda/4) = Z_0^2/Z_L$

Scattering matrix

Traveling waves: $V^+ = Ae^{-\gamma x}$, $V^- = Be^{\gamma x}$ so that the total voltage is $V(x) = V^+(x) + V^-(x)$.

Similarly for current: $I(x) = I^+(x) - I^-(x)$, $= V^+/Z_0 - V^-/Z_0$

Refl. coeff.: $\Gamma(x) = V^-(x)/V^+(x)$

Introduce “normalized” variables: $v(x) = V(x)/\sqrt{Z_0}$, $i(x) = \sqrt{Z_0}I(x)$, $a(x) = V^+(x)/\sqrt{Z_0}$, $b(x) = V^-/\sqrt{Z_0}$, so that

$v(x) = a(x) + b(x) \quad i(x) = a(x) - b(x)$ and $b(x) = \Gamma(x)a(x)$

This defines a single port network. What about 2-port?

2-port figure. Generalize eq. :

$$\begin{aligned} b_1 &= S_{11}a_1 + S_{12}a_2 \\ b_2 &= S_{21}a_1 + S_{22}a_2 \end{aligned} \tag{10}$$

or in matrix form.

Each reflected wave (b_1, b_2) has two contributions: one from the incident wave at the same port and another from the incident wave at the other port.

S-parameters, scattering parameters, scattering matrix. How to calculate them?

$$S_{11} = \left.\frac{b_1}{a_1}\right|_{a_2=0} \quad \text{input refl. coeff. with output matched} \tag{11}$$

$$S_{12} = \left.\frac{b_1}{a_2}\right|_{a_1=0} \quad \text{reverse trans. coeff. with input matched} \tag{12}$$

$$S_{21} = \left.\frac{b_2}{a_1}\right|_{a_2=0} \quad \text{transmission coeff. with output matched} \tag{13}$$

$$S_{22} = \left.\frac{b_2}{a_2}\right|_{a_1=0} \quad \text{output refl. coeff. with input matched} \tag{14}$$

"matched" = termination equal to the tr. line ch. impedance. Fig. 1.4.2.

Note: Z_{out} (of 2-port) need not be matched to Z_{02}!!

Sufficient condition: $Z_L = Z_{02} \Rightarrow a_2 = 0$. Usually $Z_{01} = Z_{02}$. We measure "overall" S-parameters, i.e. including the tr. line on I/O.

Advantage: using matched resistive terminations to measure S-matrix, transistor does not oscillate.

Occasional use for Chain scattering parameters or T-parameters — mainly for cascading networks.

Shifting reference planes

why: to "de-embed" the 2-port from tr. lines on I/O. Reference planes: positions along tr. lines (usually beginning and end). Fig. 1.5.1 (eqs by hand).

Reference planes: positions along tr. lines (beginning and end).

Fig. 1.5.1:

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \text{ and } \begin{bmatrix} b'_1 \\ b'_2 \end{bmatrix} = \begin{bmatrix} S'_{11} & S'_{12} \\ S'_{21} & S'_{22} \end{bmatrix} \begin{bmatrix} a'_1 \\ a'_2 \end{bmatrix}$$

$\Theta = \beta l$ are electrical lengths of tr. lines. Signal is delayed by Θ as it travels from $x = 0$ to $x = l \Rightarrow$

$$\left.\begin{array}{ll} b'_1 = b_1 e^{-j\Theta_1} & b'_2 = b_2 e^{-j\Theta_2} \\ a_1 = a'_1 e^{-j\Theta_1} & a_2 = a'_2 e^{-j\Theta_2} \end{array}\right\} \Rightarrow \begin{array}{ll} b_1 = b'_1 e^{j\Theta_1} & b_2 = b'_2 e^{j\Theta_2} \\ a_1 = a'_1 e^{-j\Theta_1} & a_2 = a'_2 e^{-j\Theta_2} \end{array}$$

This is plugged into first part of eq. :

$$\begin{bmatrix} b'_1 \\ b'_2 \end{bmatrix} = \begin{bmatrix} S_{11} e^{-j2\Theta_1} & S_{12} e^{-j(\Theta_1+\Theta_2)} \\ S_{21} e^{-j(\Theta_1+\Theta_2)} & S_{22} e^{-j2\Theta_2} \end{bmatrix} \begin{bmatrix} a'_1 \\ a'_2 \end{bmatrix}$$

By inspection, first term on RHS must be $= [S']$ matrix

Reverse relations obtained by expressing primed variables in eq. and plugging into second part of eqs. $\Rightarrow$

$$\begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} = \begin{bmatrix} S'_{11}e^{j2\Theta_1} & S'_{12}e^{j(\Theta_1+\Theta_2)} \\ S'_{21}e^{j(\Theta_1+\Theta_2)} & S'_{22}e^{j2\Theta_2} \end{bmatrix}$$

<u>Properties of S-parameters</u>

Start with fig. 1.6.1; assume that tr. lines are lossless and Z_0 -s are real (usually 50 Ohm lines with 50 Ohm terminations).

At i-th port (here: 1 or 2):

$$V_i(x_i) = V_i^+(x_i) + V_i^-(x_i) \tag{15}$$

$$I_i(x_i) = I_i^+(x_i) - I_i^-(x_i) = \frac{V_i^+(x_i)}{Z_{0i}} - \frac{V_i^-(x_i)}{Z_{0i}} \tag{16}$$

$$\text{define } a_i(x_i) = \frac{V_i^+(x_i)}{\sqrt{Z_{0i}}} = \sqrt{Z_{0i}} I_i^+(x_i) = \frac{1}{2\sqrt{Z_{0i}}}\left[V_i(x_i) + Z_{0i} I_i(x_i)\right] \tag{17}$$

$$\text{and similarly } b_i(x_i) = \frac{V_i^-(x_i)}{\sqrt{Z_{0i}}} = \sqrt{Z_{0i}} I_i^-(x_i) = \frac{1}{2\sqrt{Z_{0i}}}\left[V_i(x_i) - Z_{0i} I_i(x_i)\right] \tag{18}$$

V-s and I-s are peak values. For sin() signal $\Rightarrow$ divide by $\sqrt{2}$.

Average power associated with incident wave

$$P_i^+(x=0) = \frac{1}{2}\text{Re}[V_i^+(0)(I^+(0))^*] = \frac{|V_i^+(0)|^2}{2Z_{0i}} = \frac{a_i(0)a_i^*(0)}{2} = \frac{|a_i(0)|^2}{2} \quad (19)$$

and for reflected power

$$P_i^-(x=0) = \frac{|V_i^-(0)|^2}{2Z_{0i}} = \frac{b_i(0)b_i^*(0)}{2} = \frac{|b_i(0)|^2}{2} \quad (20)$$

Lines are lossless $\Rightarrow P_i^+(0) = P_i^+(l)$ and $P_i^-(0) = P_i^-(l)$

$\Rightarrow |a_i(x)|^2$ and $|b_i(x)|^2$ represent power associated with incident and reflected waves anywhere on tr. lines.

Consider Fig. 1.6.2: (port $1'$ excited by generator $E_1\angle 0^\circ$ and ports match terminated.

$$\text{At } x_2 = 0 \;:\; V_2(x_2 = 0) = -I_2(0)Z_{02},\; a_2(x_2 = 0) = 0 \quad (21)$$

$$\text{At } x_1 = 0 \;:\; V_1(x_1 = 0) = E_1 - Z_{01}I_1(0) \quad (22)$$

Resulting in (use eq. 17)

$$a_1(x_1) = \frac{E_1}{2\sqrt{Z_{01}}} \quad \text{or} \quad |a_1|^2 = \frac{|E_1|^2}{4Z_{01}} \Rightarrow P_1^+(0) = \frac{|E_1|^2}{8Z_{01}} \quad (23)$$

$\Rightarrow |a_1(x_1 = 0)|^2$ represents the power available from the source at port $1'$ and since tr. line is lossless it is also power available at port 1. Called P_{avs}. Independent of the input impedance of the 2-port network.

If $Z_1 \neq Z_{01}$, after simple derivation (see book):

$$\frac{1}{2}(|a_1(0)|^2 - |b_1(0)|^2) = \frac{1}{2}\mathrm{Re}[I_i(0)V_1^*(0)] \qquad (24)$$

which represents power delivered to port 1 (or $1'$). Call it P_1 so that $|b_1(0)|^2 = P_{avs} - P_1$ is reflected power from port 1 to port $1'$.

Similarly, $P_L' = P_L = |b_2|^2/2 = Z_{02}|I_2(0)|^2$ represents the power delivered to the load Z_{02}

Summary: Generator send available power $|a_1(0)|^2$ toward input port 1. This power is independent of input impedance Z_1. If $Z_1 = Z_{01}$ (input matched to tr. line) then reflected power =0. Otherwise, some is reflected back to generator ($= |b_1(0)|^2$). Net power delivered to port 1 is $|a_1(0)|^2 - |b_1(0)|^2$.

How to use this to calculate S-parameters? See fig. 1.6.3 and use definition of S_{11}

$$S_{11} = \left.\frac{b_1(l_1)}{a_1(l_1)}\right|_{a_2(l_2)=0} = \left.\frac{V_1^-(l_1)}{V_1^+(l_1)}\right|_{V_2^+(l_2)=0} = \frac{Z_1 - Z_{01}}{Z_1 + Z_{01}} \qquad (25)$$

S_{11} is **reflection coefficient of port 1 when port 2 is match terminated**.

Also, note that

$$|S_{11}|^2 = \left|\frac{b_1(l_1)}{a_1(l_1)}\right|^2_{a_2(l_2)=0} = \frac{P_{avs} - P_1}{P_{avs}} \qquad (26)$$

i.e. it is = ratio of power reflected from port 1 to power available at port 1.

Completely analogous situation for S_{22}, just change indices (see fig. 1.6.4)

What about S_{21} (and S_{12})? Take the definition:

$$S_{21} = \left.\frac{b_2(l_2)}{a_1(l_1)}\right|_{a_2(l_2)=0} = \left.\frac{\sqrt{Z_{02}}I_2^-(l_2)}{\sqrt{Z_{01}}I_1^+(l_1}\right|_{I_2^+(l_2)=0} \tag{27}$$

The total current at l_2 consists only of I_2^- (matching condition ensures that), i.e. $I_2(l_2) = -I_2^-(l_2)$.

$$S_{21} = \left.\frac{-\sqrt{Z_{02}}I_2(l_2)}{\sqrt{Z_{01}}I_1^+(l_1)}\right|_{I_2^+(l_2)=0} \tag{28}$$

Furthermore, $I_1^+(l_1) = E_{1,TH}/(2Z_{01})$

(why 2? $a_1(0) = E_1/(2\sqrt{Z_{01}})$
$\Rightarrow a_1(l_1) = E_1/(2\sqrt{Z_{01}})\exp(-j\Theta_1)$.

$Z_{TH} = Z_{01}$ (matched line), $E_{TH} = |E_1|\angle -\Theta_1$. $\Rightarrow a_1(l_1) = \sqrt{Z_{01}}I_1^+(l_1) = E_{TH}/(2\sqrt{Z_{01}}) \Rightarrow I_1^+(l_1) = E_{TH}/(2Z_{01})$)

Also: $V_2(l_2) = -Z_{02}I_2(l_2)$. $\Rightarrow$

$$S_{21} = \frac{\sqrt{Z_{02}}V_2(l_2)/Z_{02}}{\sqrt{Z_{01}}E_{1,TH}/(2Z_{01})} = 2\sqrt{\frac{Z_{01}}{Z_{02}}}\,\frac{V_2(l_2)}{E_{1,TH}} \tag{29}$$

$\Rightarrow S_{21}$ is **forward voltage transmission coefficient from port 1 to port 2**.

Also:

$$|S_{21}|^2 = \frac{\frac{1}{2}|V_2(l_2)|^2/Z_{02}}{|E_{1,TH}|^2/(8Z_{01})} = \frac{\text{P delivered to load}}{P_{avs}} \tag{30}$$

i.e. $|S_{21}|^2$ is **transducer power gain** (note the loading conditions!).

S_{12} analogous — change the indices.

Do example 1.6.1 for S-param. calculation and another ex.

$S_{11} = b_1(l_1)/a_1(l_1)$ $(a_2(l_2) = 0)$ is input reflection coeff. with output match terminated (tr. line matching condition)

$|S_{11}|^2 = (P_{avs} - P_1)/P_{avs} =$ ratio of power reflected from port 1 to power available from source

$S_{21} = 2\sqrt{Z_{01}/Z_{02}} V_2(l_2)/E_{1,TH}$ is forward transmission coeff. from port1 to port2 with output matched

$|S_{21}|^2 = P_L/P_{avs}$ is ratio of power delivered to load Z_{02} to available power = transducer power gain

For S_{22} and S_{12} just interchange ports (terminals).

Generalized S-parameters

So far: 50 Ohm tr. lines and 50 Ohm terminations. How about n-port networks with general terminations?

$$V_i(x_i) = V_i^+(x_i) + V_i^-(x_i) \tag{31}$$

$$I_i(x_i) = I_i^+(x_i) - I_i^-(x_i) = \frac{V_i^+(x_i)}{Z_{0i}} - \frac{V_i^-(x_i)}{Z_{0i}} \tag{32}$$

define $a_i(x_i) = \dfrac{V_i^+(x_i)}{\sqrt{Z_{0i}}} = \sqrt{Z_{0i}} I_i^+(x_i)$

$$= \frac{1}{2\sqrt{Z_{0i}}}[V_i(x_i) + Z_{0i} I_i(x_i)] \tag{33}$$

and similarly $b_i(x_i) = \dfrac{V_i^-(x_i)}{\sqrt{Z_{0i}}} = \sqrt{Z_{0i}} I_i^-(x_i)$

$$= \frac{1}{2\sqrt{Z_{0i}}}[V_i(x_i) - Z_{0i} I_i(x_i)] \tag{34}$$

how to generalize this for any impedance?

$$[a] = \frac{1}{2}\left[\frac{1}{\sqrt{R_{0,i}}}\right]\left([V] + [Z_{0,i}]\,[I]\right) \tag{35}$$

$$[b] = \frac{1}{2}\left[\frac{1}{\sqrt{R_{0,i}}}\right]\left([V] - [Z_{0,i}]^{*}\,[I]\right) \tag{36}$$

$$[Z_{0,i}] = \begin{bmatrix} Z_{0,1} & 0 & \cdots & 0 \\ 0 & Z_{0,2} & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & Z_{0,n} \end{bmatrix}$$

$$\left[R_{0,i}^{-1/2}\right] = \begin{bmatrix} (\mathsf{Re}(Z_{0,1}))^{-1/2} & 0 & \cdots & 0 \\ 0 & (\mathsf{Re}(Z_{0,2}))^{-1/2} & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & (\mathsf{Re}(Z_{0,n}))^{-1/2} \end{bmatrix}$$

$$a_1 = \frac{1}{2} R_{0,1}^{-1/2} (V_1 + Z_{0,1} I_1) \tag{37}$$

$$b_1 = \frac{1}{2} R_{0,1}^{-1/2} (V_1 - Z_{0,1}^{*} I_1) \tag{38}$$

As before, expressing V_1 gives

$$V_1 = E_1 - Z_{G1} I_1 \Rightarrow a_1 = \frac{E_1}{2R_{0,1}^{1/2}} \Rightarrow |a_1|^2 = \frac{|E_1|^2}{4R_{0,1}} \tag{39}$$

$=$ power available from source at port 1.

Also (as before)

$$|a_1|^2 - |b_1|^2 = \mathrm{Re}(V_1^* I_1) \tag{40}$$

$=$ power delivered to port 1.

If port 1 is matched (so that $Z_1 = V_1/I_1 = Z_{0,1}^*$), power $|a_1|^2$ is completely absorbed by Z_1. If not matched, then power absorbed is $|a_1|^2 - |b_1|^2$.

Generalized scattering matrix

Define it via $[b] = [S][a]$. S-matrix depends on the choice of normalizing impedance. Usually 50 Ohms, but can be anything and can even be complex.

Calculating S_{ij}:

$$S_{ii} = \left.\frac{b_i}{a_i}\right|_{a_k=0, k\neq i, k=1,\ldots,n} = \frac{V_i - Z_{0,i}^* I_i}{V_i + Z_{0,i}^* I_i} = \frac{Z_i - Z_{0,i}^*}{Z_i + Z_{0,i}} \tag{41}$$

which is input reflection coeff. with all other ports matched.

$$S_{ki} = \left.\frac{b_k}{a_i}\right|_{a_k=0, k\neq i, k=1,\ldots,n} \tag{42}$$

$=$ transducer power gain from i to k with ports other than i matched.

Parameter conversions: see tables. They all describe the same thing! Note that conversion to and from S-parameters is dependent on choice of Z_0 .

S-parameters for transistors

Most common way of specifying transistor performance. Usually CE but can be converted to CB. CE $\rightarrow$ CB conversion of S-parameters has to go through y or z-parameters. $S_{CE} \rightarrow Y_{CE} \rightarrow Y_{CB} \rightarrow S_{CB}$

Simple R-C circuit S-chart in fig 1.9.1

For S-chart presentation see fig. 1.9.2. S_{11} and S_{22} typically behave as RLC circuits (fig. 1.9.3).

S_{21} and S_{12} usually presented on polar plot. Interesting to see the magnitudes vs. freq. in fig. 1.9.7. $|S_{21}|$ usually exhibits 6 db/octave fall-off. f_s is its cut-off frequency.

S_{12} exhibits opposite behavior (why?).

If no choice of ground terminal has been made $\Rightarrow$ indefinite scattering matrix. Transistor treated as 3-port. This is a different matrix than the one discussed before, e.g.:

$$S_{11} = \left.\frac{b_1}{a_1}\right|_{a_2=0, a_3=0} \tag{43}$$

Note: sum of the rows and columns in indefinite S-matrix is unity! $\Rightarrow$ no need to measure the third one.