Uncertainty Estimation and Calculation

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Version 0.48   September 11, 2012

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Overview

- Phases of Experimental Measurement
- Types of Errors
- Measurement chain
- Estimating the true value of a measured quantity
- Estimating uncertainties

Phases of an Experiment

1. Planning
2. Design
3. Fabrication
4. Shakedown
5. Data collection and analysis
6. Reporting

Uncertainty analysis is very useful in the Design phase. It should be considered mandatory in the Data collection and analysis phase.

The discussion of uncertainty analysis in these notes is focused on the data collection and analysis phase.

Start with the Truth

- Goal is to experimentally measure a physical quantity

- The *true value of the quantity is a concept.* In almost all cases, the true value cannot be measured.

- The *error* in a measurement is the difference between the true value and the value reported as a result of a measurement. If x is the quantity of interest

$$\text{Error} = x_{\text{measured}} - x_{\text{true}}$$

- A claim of numerical *Accuracy* establishes an upper bound on the error.

x is accurate to
within 5 percent
$\implies$
Maximum expected value of
$|x_{\text{measured}} - x_{\text{true}}|$ is $0.05|x_{\text{true}}|$

- A numerical value of *Uncertainty is an estimate of the error.* The *uncertainty quantifies the expected accuracy*, but *it is not a guarantee of accuracy*.

Role of the True Value

The *true value* of an object being measured is rarely ever known.

1. Identifying the true value needed by an end user is critical. What are we trying to measure?
2. The true value is often a concept that is very hard (or impossible) to measure.

Moffat [3] uses the example of average surface temperature as a true value that is needed for defining the heat transfer coefficient. Measurement of an average surface temperature is impossible because of the huge number of sensors that would be needed.

- A large number of sensors would cause a large disturbance to the system.
- True surface temperature must be area-weighted. For convection the appropriate area-weighting is $(1/A) \int T \, dA$. For radiation the appropriate area-weighting is $\left[(1/A) \int T^4 \, dA \right]^{1/4}$

So even the same apparatus could require two (or more) true values to define the same concept.

Examining Errors

Our Goal:

Obtain an estimate of the uncertainty in measured results

The Plan:

1. Classify errors
2. Identify sources of errors
3. Remove errors we can correct
4. Develop a procedure for computing uncertainty
5. Demonstrate how to apply these methods to flow rate measurement.

References:

For additional information on uncertainty analysis and errors measurements see [1, 3, 2, 4, 5] The following lecture notes are primarily based on Moffat [3].

Classifying Errors (1)

Ideal Distinction: *bias* versus *random* errors

Bias error is a systematic inaccuracy caused by a mechanism that we can (ideally) control. We might be able to adjust the way measurement are taken in an attempt to reduce bias errors. We can try to correct bias errors by including adjustments in our data analysis *after* the measurements are taken.

Random error is a non-repeatable inaccuracy caused by an unknown or an uncontrollable influence. Random errors introduce scatter in the measured values, and propagate through the data analysis to produce scatter in values computed from the measurements. Ideally random errors establish the limits on the precision of a measurement, not on the accuracy of a measurement.

Classifying Errors (2)

Moffat [3, § 2.2.1] makes a more practical distinction of three types of errors.

- Fixed errors
- Random errors
- Variable but deterministic errors

In Moffat's taxonomy, fixed errors and variable but deterministic errors are bias errors.

Fixed Error

All repeatable errors are fixed errors. A fixed error is the same for each nominal operating point of the system

Sources of fixed error

- sensor calibration
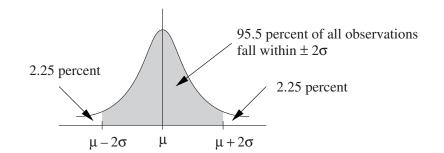- non-random disturbance to system

Examples

- A pressure gauge that always reads 2 psi high at 100 psi.
- Heat flow along thermocouple leads when measuring the temperature of an object
- Effect of probe blockage on flow field downstream of the probe

Random Error

Random errors have different magnitude during subsequent measurements of the same quantity. For well behaved systems, random errors in a measurement cause a cluster of observed values.

- We will assume that random errors are normally distributed. This is reasonable for large sample sizes and truly random errors

- A large sample (say, 30 or more observations) of the system *at the same operating point* tends toward a normal (Gaussian) distribution of values about some mean value if the population is Gaussian.

Variable but Deterministic Error (1)

Some errors that appear to be random can be caused by faulty measurement techniques (e.g. aliasing) or the errors may be *variable but deterministic.*
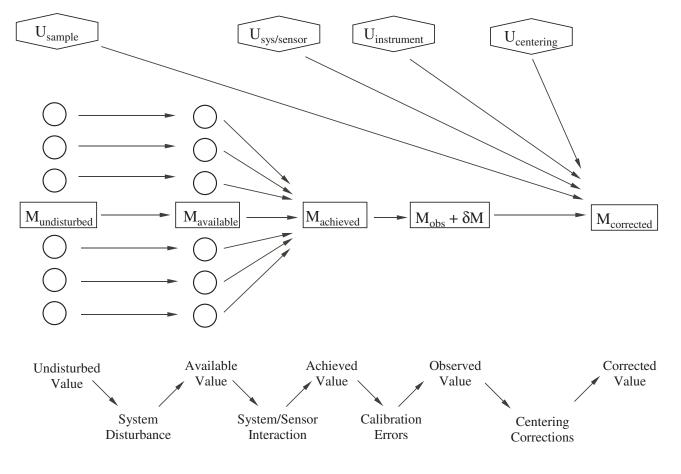
- Errors change even though the system is at the same nominal operating point
- Errors may not be recognized as deterministic: variations between tests, or test conditions, may seem random.
- Cause of these errors are initially hidden from the experimenter

Variable but Deterministic Error (2)

Examples:

- Variations in room air conditions such as temperature and air circulation patterns
  - ▷ set-back thermostats
  - ▷ solar radiation through windows
  - ▷ presence of people in the room
  - ▷ windows open to outside

- Changes in sensors
  - ▷ Thermal drift of sensors
  - ▷ Use of a new batch of thermocouple wire with different calibration
  - ▷ Cold working of thermocouple wire

- Changes in consumable materials or equipment used in experiments
  - ▷ Leakage or chemical degradation of working fluid
  - ▷ Mechanical wear or misalignment of positioning equipment

Moffat's Measurement Chain

Adapted from R.J. Moffat, " Uncertainty Analysis" , Chapter 2, in **Thermal Measurements in Electronic Cooling**, Kaveh Azar, ed., 1997, CRC Press, Boca Raton, FL.

Values of Measurand Along the Chain (1)

$$\boxed{\text{M}_{\text{undisturbed}}} \longrightarrow \cdots \longrightarrow \boxed{\text{M}_{\text{corrected}}}$$

1. **Undisturbed value**: The value of quantity to be measured in a system with no sensors.

   Example: Temperature at point (x, y, z) is $T(x, y, z)$.

2. **Available value**: The value of quantity to be measured *after* the sensor has been placed in the system.

   Example: Temperature at point (x, y, z) after sensor is been inserted is $T_A(x, y, z) \neq T(x, y, z)$.

3. **Achieved value**: The value of quantity to be measured *actually experienced* by the sensor.

   Example: Temperature at the junction of a thermocouple at point (x, y, z) is $T_j(x, y, z) \neq T_A(x, y, z)$.

Values of Measurand Along the Chain (2)

4. **Observed value**: The value indicated by the instrumentation.

    Example: EMF produced by thermocouple with temperature at $T_j(x, y, z)$ is $E_j(x, y, z)$. The temperature indicated by the thermocouple is $T_i = f(E_j)$, and $T_i \neq T_j(x, y, z) \neq T_A(x, y, z) \neq T(x, y, z)$.

5. **Corrected value**: The value obtained after making centering corrections to the observed value.

    Example: $T_c = T_i + \Delta T$ where ΔT is a centering correction to account primarily for the difference $T_j - T_A$. We design a model for the centering correction so that T_c is closer to T than T_i.

Contributions to Uncertainty of Measurand (1)

$$\boxed{U_{sample}} \quad \cdots \quad \boxed{U_{centering}}$$

1. **Sampling Uncertainty**

   The undisturbed value is one of a large population of possible values for the undisturbed quantity of interest.

   Sampling Uncertainty is due to random errors.

2. **System Sensor Uncertainty**
   - System disturbance changes the state of the system
   - The sensor experiences interactions with the environment, e.g., radiation and conduction effects on thermocouple measurement.

   System Sensor Uncertainty is due to a fixed error.

Contributions to Uncertainty of Measurand (2)

3. **Instrument Uncertainties**

   Estimates of instrumentation uncertainties are readily made, though not all such estimates reflect the true uncertainty. Moffat claims that with modern instrumentation, the instrument uncertainties are often the least significant.

   Instrument Uncertainties are due to random errors *for a properly calibrated instrument*.

4. **Centering Uncertainties**

   Each centering correction has an associated uncertainty. These must be included in a final estimate of the uncertainty in the value of the measured quantity.

   Centering Uncertainties are due to fixed errors. We accept these fixed errors *after* we have done our best to minimize them.

Estimating Uncertainties (1)

1. **System Disturbance Errors:** The presence of a sensor changes the system
   - Can be estimated, but not with any certainty
   - Use estimates of disturbance errors to design the experiment, and to inform the installation of sensors in the experimental apparatus. For example, locate a probe to minimize flow blockage, or leakage.
   - Goal is to design, build and instrument the system to minimize system disturbance errors.

2. **System/Sensor Interaction Errors:** System phenomena alter value sensed by the sensor.

   Example: Radiation from the walls of a duct alter the temperature of an exposed thermocouple junction used to measure the temperature of a flowing gas.
   - Should be estimated using engineering models of heat transfer and fluid flow
   - Uncertainty in the estimate of this error needs to be included in final uncertainty estimate for the quantity of interest.

Estimating Uncertainties (2)

3. **Instrumentation Errors:**
   - Use manufacturer's data or traceable calibration.
   - Best solution is to perform simple benchmark experiments to *measure the instrument errors in situ*.
   - See Figure 4, page 61 in the article by Moffat [3] for a detailed worksheet used to estimate the instrumentation error. Note that this worksheet does not included system disturbance, system/sensor interactions, or conceptual errors.

4. **Uncertainties in Centering Corrections:**
   - Report corrected values!
   - Uncertainty in the correction should be smaller than the correction itself.
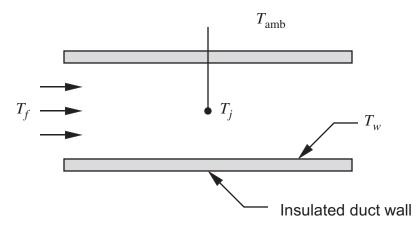
Estimating Uncertainties (3)

5. **Conceptual Errors:**
   - Can be very large
   - Not subjected to laws of physics
   - Minimize by using good practice
     - ▷ Check units.
     - ▷ Use shakedown experiments to compare results to established benchmarks.
     - ▷ Use redundant measurements, e.g. energy balances on *both sides* of an heat exchanger.
     - ▷ Publish results with sufficient information so that others can replicate your results.

6. **Overall Errors**
   - Combine with methods presented below
   - Use root-mean-squared procedure.

Example of Centering Correction (1)

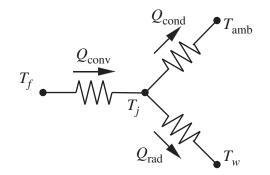**Note:** Example: Measurement of fluid temperature in a duct.

The junction exchanges heat with its surroundings by conduction, convection and radiation. Is $T_j = T_f$?

Example of Centering Correction (2)

$$Q_{\text{cond}} = \frac{k_{\text{eff}} A_{\text{eff}}}{L}(T_j - T_{\text{amb}})$$ Heat loss by conduction from junction to ambient

$$Q_{\text{rad}} = \epsilon_j \sigma A_s (T_j^4 - T_w^4)$$ Heat loss by radiation from junction to walls

$$Q_{\text{conv}} = h A_s (T_f - T_j)$$ Heat gain by convection from fluid to junction

Example of Centering Correction (3)

Steady state energy balance: Heat gain = heat loss

$$Q_{\text{conv}} = Q_{\text{cond}} + Q_{\text{rad}}$$

For simplicity we will neglect radiation. This is a good assumption if $T_w \approx T_j$.

$$Q_{\text{conv}} = Q_{\text{cond}} \implies \frac{k_{\text{eff}} A_{\text{eff}}}{L}(T_j - T_{\text{amb}}) = h A_s (T_f - T_j)$$

Solve for T_j

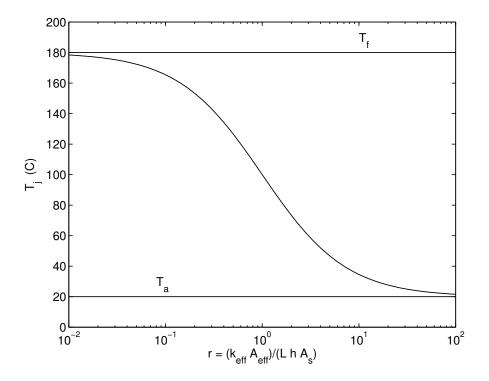$$T_j = \frac{T_f + \dfrac{k_{\text{eff}} A_{\text{eff}}}{L h A_s} T_{\text{amb}}}{1 + \dfrac{k_{\text{eff}} A_{\text{eff}}}{L h A_s}}$$

If conductivity or cross-section of the probe is large, then T_j is closer to T_{amb} than to T_f.

Example of Centering Correction (4)

Let $T_{\mathrm{amb}} = 20\,°\mathrm{C}$ and $T_f = 180\,°\mathrm{C}$.

Compute T_j for a range of r where

$$r = \frac{k_{\mathrm{eff}} A_{\mathrm{eff}}}{L h A_s}.$$

Plot shows $T_j = f(r)$ when radiation is neglected.

Example of Centering Correction (5)

Use the estimate of the conduction loss to computed a corrected value of T_f from the *measured* value of T_j.

$$T_{f,\text{corrected}} = T_j + \frac{k_{\text{eff}} A_{\text{eff}}}{L h A_s}(T_j - T_{\text{amb}})$$

In words:

T_j is the *observed* value of the sensor.

$T_{f,\text{corrected}}$ is the estimate of the true value of the fluid temperature after correcting the observed value for the effect of heat loss by conduction in the probe.

Computing Uncertainty of Measurement

1. Definitions

2. Root-sum-squared Computation

3. Numerical approximation to sensitivities

4. Sequential perturbation method

Definitions

1. Let x be the value of a measured quantity.

2. Let u_x be the uncertainty associated with x.

3. When we write
$$x = x_{\mathrm{measured}} \pm u_x \qquad (20{:}1)$$
we mean that
   - x is the best estimate of the measure value,
   - x_{measured} is the value of x obtained by correction (centering) of the measured value of x,
   - u_x is the uncertainty at stated odds. Usually 20:1 odds

Meaning of 20:1 odds

- In general, if x is a normally distributed random variable, there is a 95 percent[1] probability that any one sample of x will lie within $\pm\, 2\sigma$ of the mean of the population of all possible values of x. Thus, 19 out of 20 samples ($19/20 = 0.95$) from the population will lie within $\pm\, 2\sigma$. *Or* only 1 out of 20 samples will lie outside of the $\pm\, 2\sigma$ band about the mean.

- When we say that $x = x_\mathrm{measured} \pm u_x$ at 20:1 odds, we mean that upon repeating our experiment to measure x twenty times, we expect only one sample of x to deviate from our best estimate by more than u_x.

*Therefore u_x is *not* an absolute upper bound on the error.*

---

[1]95.45 percent, to be more precise. The 95 percent confidence interval is $\pm 1.96\sigma$.

Computational Procedures (1)

There are three ways to compute the numerical value of the uncertainty.

1. Analytical method
2. Sequential perturbation
3. Monte Carlo

The first two methods are based on the *Root-Sum-Squared* estimate of the error.

The Monte Carlo method involves repeating the data reduction procedure a large number of times (say, 1000 or 10,000 or more) for each set of input (measured) values. For each of the repeated runs, a pseudo-random value for each nominal input is chosen from a probability distribution appropriate for that input.

Computational Procedures (1)

The analytical method involves deriving a single formula for the uncertainty in a measurement.

- Straightforward computation
- Becomes unwieldy and eventually impractical as the data reduction procedure becomes increasingly complex.

The sequential perturbation technique is easy to implement when the data reduction procedure is automated via a computer program.

- Uncertainty estimate is approximate, not exact as in the analytical method.
- Is simple to implement, and allows for evolution of the *model* underlying the data reduction.

Root-Sum-Squared (RSS) Computation (1)

Begin with the basic definition of the Root-Sum-Squared uncertainty.

The data reduction computation is represented by

$$R = f(x_1, x_2, \ldots, x_n)$$

where R is the final, measured *result*, and x_i are measured quantities. Let u_i be the uncertainty in x_i so that

$$x_i = x_{\text{measured},i} \pm u_i$$

This is a slight change in notation to avoid the double subscript u_{x_i}

Root-Sum-Squared (RSS) Computation (2)

The uncertainty in R due to the uncertainties in each x_i is

$$u_R = \left[\left(\frac{\partial R}{\partial x_1} u_1 \right)^2 + \left(\frac{\partial R}{\partial x_2} u_2 \right)^2 + \ldots + \left(\frac{\partial R}{\partial x_n} u_n \right)^2 \right]^{1/2} \tag{1}$$

For simple data reduction steps its is possible to compute the derivatives analytically. For even modestly complicated data reduction processes, it makes more sense to use a computer program to perform the data reduction *and* uncertainty calculation

Example of RSS Computation (1)

Nozzle flow equation

$$Q = C_d A_n Y \sqrt{\frac{2\Delta p}{\rho(1 - \beta^4)}} \qquad (2)$$

C_d is the nozzle discharge coefficient,

A_n is the area of the nozzle exit, $A_n = (\pi/4)d_n^2$

d_n is the throat diameter,

Δp is the measured pressure drop across the nozzle,

Y is the expansion factor, which accounts for the compressibility of the fluid.

ρ is the fluid density upstream of the nozzle

$\beta = d_n/D$ is the contraction ratio,

D is the diameter of the upstream duct.

Example of RSS Computation (2)

The nozzles are built to ASME/ANSI specification, but are not individually calibrated. Use the generic equation for the discharge coefficient

$$C_d = 0.9986 - \frac{7.006}{\sqrt{\text{Re}_n}} + \frac{134.6}{\text{Re}_n} \tag{3}$$

$$\text{Re}_n = \frac{V_n d_n}{\nu} = \frac{4Q}{\pi d_n \nu} \tag{4}$$

$$Y = \left[\frac{\gamma}{\gamma - 1} \alpha^{2/\gamma} \frac{1 - \alpha^{(\gamma - 1)/\gamma}}{1 - \alpha} \right]^{1/2} \left[\frac{1 - \beta^4}{1 - \beta^4 \alpha^{2/\gamma}} \right]^{1/2} \tag{5}$$

Example of RSS Computation (3)

$$V_n = Q/A_n \quad \text{is the average velocity in the nozzle throat,}$$

$\rho \quad$ is the fluid density,

$\mu \quad$ is the fluid viscosity evaluated at the pressure and temperature upstream of the nozzle.

$$\alpha \quad = \frac{p - \Delta p}{p}$$

Example of RSS Computation (4)

Focus on Equation (2)

$$Q = C_d A_n Y \sqrt{\frac{2\Delta p}{\rho(1 - \beta^4)}} \tag{2}$$

To begin, assume that C_d, A_n, Y, and β are known.

Thus, we are (initially) considering just the uncertainty in Q due to the uncertainty in Δp and ρ.

$$u_Q = \left[\left(\frac{\partial Q}{\partial(\Delta p)} u_{\Delta p} \right)^2 + \left(\frac{\partial Q}{\partial \rho} u_\rho \right)^2 \right]^{1/2} \tag{6}$$

Example of RSS Computation (5)

Compute:

$$\frac{\partial Q}{\partial (\Delta p)} = \frac{C_d A_n Y \sqrt{2}}{\sqrt{\rho (1 - \beta^4)}} \frac{1}{2} (\Delta p)^{-1/2} \tag{7}$$

$$\frac{\partial Q}{\partial \rho} = C_d A_n Y \sqrt{\frac{2\Delta p}{(1 - \beta^4)}} \left(-\frac{1}{2}\right) \rho^{-3/2} \tag{8}$$

Substitute Equations (7) and (8) into Equation (6) and simplify to get

$$\frac{u_Q}{Q} = \left[\left(\frac{u_{\Delta p}}{2\Delta p}\right)^2 + \left(\frac{u_\rho}{2\rho}\right)^2 \right]^{1/2} \tag{9}$$

Example of RSS Computation (6)

Problems:

- u_ρ depends on measured p and T. If we include the effects of uncertainties in p and T on the value of u_ρ, then we should also include its effect on the computation of $C_d = f(\mathrm{Re}_n)$.
- What about uncertainty in d_n? This affects the values of C_d and A_n.
- Don't forget uncertainty in Y, which is determined by the values of p and Δp.
- Furthermore, we have only considered uncertainties in the property values p, T, and ρ. What about the uncertainty in the sensors used to measure p, T, and ρ?

In short: *Equation (9) is an incomplete model of u_Q.*

Assessment of RSS Computation

- Analytical technique is simple in concept, but cumbersome to implement for measurement results (R) that depend on many individual measurements (x_i).

- Random errors can not be easily included in individual measurements because random errors from sensor readings need to be combined with calibration errors.

- Including uncertainty in sensors requires impractical level of analytical detail.

- The formula for the uncertainty estimates must be re-derived if any step in the measurement chain changes.

- Derivation of the uncertainty formula will be incorrect if there are any errors in the algebra.

Fortunately *there is an easier way!*

Perturbation Method for Computing Uncertainties (1)

As a consequence of the Fundamental Theorem of Calculus

$$\frac{\partial R}{\partial x_i} = \lim_{\Delta x_i \to 0} \left[\frac{R(x_i + \Delta x_i) - R(x_i)}{\Delta x_i} \right] \approx \frac{R(x_i + \delta x_i) - R(x_i)}{\delta x_i}$$

where δx_i is a finite perturbation in the measured value of x_i.

Now, *if we use $\delta x_i \approx u_i$*, then

$$\left[\frac{\partial R}{\partial x_i} \delta x_i \right]^2 \approx \left[\frac{R(x_i + \delta x_i) - R(x_i)}{\delta x_i} u_i \right]^2 \approx \left[R(x_i + u_i) - R(x_i) \right]^2$$

In words: The uncertainty in R due to the uncertainty in x_i can be estimated by perturbing the data reduction formula by u_i.

Perturbation Method for Computing Uncertainties (2)

Equation (1) becomes

$$u_R = \left[\mathcal{D}_1^2 + \mathcal{D}_2^2 + \ldots + \mathcal{D}_n^2 \right]^{1/2} \qquad (10)$$

where

$$\mathcal{D}_1 = R(x_1 + u_1, x_2, \ldots, x_n) - R(x_1, x_2, \ldots, x_n)$$

$$\mathcal{D}_2 = R(x_1, x_2 + u_2, \ldots, x_n) - R(x_1, x_2, \ldots, x_n)$$

$$\vdots$$

$$\mathcal{D}_n = R(x_1, x_2, \ldots, x_n + u_n) - R(x_1, x_2, \ldots, x_n)$$

The uncertainty in R is estimated by *sequentially* perturbing the input values x_i by their respective uncertainty.

Perturbation Method for Computing Uncertainties (3)

Equation (10)

$$u_R = \left[\mathcal{D}_1^2 + \mathcal{D}_2^2 + \ldots + \mathcal{D}_n^2 \right]^{1/2} \tag{10}$$

is valid for small perturbations, u_i. If $R(x_1, x_2, \ldots, x_n)$ is strongly nonlinear, or if any of the u_i are large, then the u_R computed from Equation (10) may not be a good approximation to the u_R computed from Equation (1).

Implementation of the Perturbation Method

The preceding idea is best implemented in a computer program.

1. Write a general subroutine to evaluate $R(x_1, x_2, \ldots, x_n)$, where x_i are the measured inputs.

2. Obtain estimates of u_i:
   - Contribution from calibration errors
   - Contribution from random variation of inputs: use statistical sampling

3. Call the general subroutine once to get the *nominal* value of $R(x_1, x_2, \ldots, x_n)$

4. For each x_i compute

$$\mathcal{D}_i = R(x_1, x_2, \ldots, x_i + u_i, \ldots, x_n) - R(x_1, x_2, \ldots, x_i, \ldots, x_n)$$

5. Combine contributions to the uncertainty with

$$u_R = \left[\mathcal{D}_1^2 + \mathcal{D}_2^2 + \ldots + \mathcal{D}_n^2 \right]^{1/2}$$

Example: Nozzle Flow Rate Calculation (1)

Use sequential perturbation to compute the uncertainty of the flow through a nozzle. Recall Equation (2).

$$Q = C_d A_n Y \sqrt{\frac{2\Delta p}{\rho(1-\beta^4)}} \qquad (2)$$

Rewrite as

$$Q = f(d_n, D, p_n, T_1, p_1) \qquad (11)$$

Recall that $\beta = d_n/D$, where D is the diameter of the upstream plenum.

Consider the influence of individual uncertainties u_{d_n}, u_D, $u_{\Delta p}$, u_{T_1}, u_{p_1} on the value of Q.

Let Q_{nom} be the nominal value of the flow rate, i.e., Q_{nom} is the flow rate for the measured set of $(d_n, D, \Delta p, T_1, p_1)$ values.

Example: Nozzle Flow Rate Calculation (2)

Sequential perturbation:

$$\mathcal{D}_{dn} = f(d_n + u_{dn}, D, \Delta p, T_1, p_1) - Q_{\text{nom}}$$

$$\mathcal{D}_D = f(d_n, D + u_D, \Delta p_n, T_1, p_1) - Q_{\text{nom}}$$

$$\mathcal{D}_{pn} = f(d_n, D, \Delta p + u_{\Delta p}, T_1, p_1) - Q_{\text{nom}}$$

$$\mathcal{D}_{T_1} = f(d_n, D, \Delta p, T_1 + u_{T_1}, p_1) - Q_{\text{nom}}$$

$$\mathcal{D}_{p_1} = f(d_n, D, \Delta p, T_1, p_1 + u_{p_1}) - Q_{\text{nom}}$$

Then

$$u_Q = \left[(\mathcal{D}_{dn})^2 + (\mathcal{D}_D)^2 + (\mathcal{D}_{\Delta p})^2 + (\mathcal{D}_{T_1})^2 + (\mathcal{D}_{p_1})^2 \right]^{1/2}$$

The calculations are demonstrated in the `demoUncertainty` m-file and the `flowBenchUncertainty` Excel spreadsheet.

Combining Uncertainties

Up to this point we've considered the effect of a given uncertainty on a computed result. How do we obtain the estimate of the uncertainty in a primary measured value?

In particular, how do we account for the uncertainty in a single measured value when the following error sources contribute to the uncertainty?

1. Random errors, $u_{x_{\text{rand}}}$
2. Instrumentation errors, $u_{x_{\text{inst}}}$
3. Calibration errors, $u_{x_{\text{cal}}}$

Assuming that the individual contributions are uncorrelated:

$$u_x = \sqrt{u_{x_{\text{rand}}}^2 + u_{x_{\text{inst}}}^2 + u_{x_{\text{cal}}}^2}$$

Example: T-couple Measurement with Hand-Held Multimeter (1)

The smallest resolution is $0.1\,\text{mV}$, so the uncertainty due to the instrument resolution is $\pm 0.05\,\text{mV}$. This does not mean that the accuracy of the instrument is $\pm 0.05\,\text{mV}$.

At $20\,^\circ\text{C}$, the EMF of an ice-point referenced, type T thermocouple is $0.7896\,\text{mV}$

$$\text{EMF} + u_V = 0.7896 + 0.05 = 0.8396\,\text{mV}$$

$$\implies T = 21.2\,^\circ\text{C}$$

$$\text{EMF} - u_V = 0.7896 - 0.05 = 0.7396\,\text{mV}$$

$$\implies T = 18.8\,^\circ\text{C}$$

Therefore, the uncertainty in temperature due to the resolution of the instrumentation is

$$u_{T_{\text{inst}}} = \pm\frac{21.2 - 18.8}{2} = \pm\,1.2\,^\circ\text{C}$$

Example: T-couple Measurement with Hand-Held Multimeter (2)

Summary of uncertainties

$$u_{T_{\text{rand}}} = 0\,°\text{C} \qquad \text{No observed fluctuations}$$

$$u_{T_{\text{inst}}} = \pm 1.2\,°\text{C} \qquad \text{See calculation above}$$

$$u_{T_{\text{cal}}} = 0.1\,°\text{C} \qquad \text{Maximum error indicated on the thermocouple}$$
$$\text{spool: } 0.2\,°\text{F at } 200\,°\text{F}$$

Combining the uncertainties gives

$$u_T = \sqrt{1.2^2 + 0.1^2} = 1.20\,°\text{C}$$

which is the overall uncertainty in the temperature measurement for an ice-point referenced thermocouple read with the hand-held multimeter with voltage resolution of $\pm 0.05\,\text{mV}$.

Example: T-couple Measurement with Hand-Held Multimeter (3)

Summary of uncertainties (repeat from previous slide)

$$u_{T_{\mathrm{rand}}} = 0\,°\mathrm{C} \qquad \text{No observed fluctuations}$$

$$u_{T_{\mathrm{inst}}} = \pm 1.2\,°\mathrm{C} \qquad \text{See calculation above}$$

$$u_{T_{\mathrm{cal}}} = 0.1\,°\mathrm{C} \qquad \text{Maximum error indicated on the thermocouple}$$
$$\text{spool: } 0.2\,°\mathrm{F} \text{ at } 200\,°\mathrm{F}$$

For the assumed values in this example, the measurement error due to the inherent properties of the thermocouple wire are negligible.

The uncertainty in the measurement (for this example!) is due to the limited number of significant digits available on the electronic measurement instrument.

Example: T-couple Measurement with Scanning Multimeter (1)

Repeat the thermocouple measurement with an Agilent$^{\text{TM}}$ data acquisition and switch unit.

The manufacturer's claimed accuracy is $\pm 0.005 \, \mu\text{V}$ with one hour warm up and 6 and 1/2 digit readings. Although our readings were taken with 5 and 1/2 digit setting, we can assume that the error in voltage measurement introduced by this data acquisition system is zero.

Therefore, for the Agilent multimeter

$$u_{T_{\text{inst}}} = 0$$

Note: The statement $u_{T_{\text{inst}}} = 0$ means that *compared to other sources of uncertainty*, the uncertainties due to the electrical calibration and precision of the Agilent multimeter are negligible.

Example: T-couple Measurement with Scanning Multimeter (2)

Suppose that as the result of 30 samples of an ice-point referenced thermocouple, the reading is $\bar{E} = 0.831\,\text{mV}$ and $\sigma_E = 3.06\,\mu\text{V}$. Then the uncertainty due to the fluctuations in DC voltage alone is

$$u_{T_{\text{rand}}} = G(0.831 + 2(0.0036)\,\text{mV}) - G(0.831\,\text{mV})$$
$$= 0.178\,^{\circ}\text{C}$$

where $G(E)$ evaluates the calibration polynomial for the thermocouple. The factor of two multiplying σ_E term is necessary to be consistent with the 95 percent confidence interval (20:1 odds).

The preceding calculation can be done with the `Ttemp` MATLAB function:

```
>> uTrand = Ttemp( (0.831 + 2*(0.0036))*1e-3 ) - Ttemp(0.831e-3)
uTrand =
    0.1782
```

Example: T-couple Measurement with Scanning Multimeter (3)

The calibration error for the thermocouple wire is $\pm 0.1\,°\mathrm{C}$.

Finally, we combine the uncertainties due to random fluctuations in the signal, and the uncertainty due to the wire calibration.

$$u_T = \sqrt{u_{T_{\mathrm{rand}}}^2 + u_{T_{\mathrm{cal}}}^2} \qquad (u_{T_{\mathrm{inst}}} = 0)$$

$$= \sqrt{0.178^2 + 0.1^2} = 0.20\,°\mathrm{C}$$

This is the overall uncertainty in the temperature measurement for an ice-point referenced thermocouple read with the Agilent multimeter.

The uncertainty in the temperature measurement with the Agilent multimeter is an order of magnitude less than the uncertainty in the temperature measurement with the hand-held multimeter. This result is primarily due to the much higher resolution of the Agilent multimeter. Note that the calculations of uncertainty assumed that there was no calibration errors in these instruments.

Example: T-couple Measurement with Scanning Multimeter (4)

What if the thermocouple is referenced to the zone box temperature and the temperature of the zone box is measured with a thermistor?

Now, two streams of uncertainties contribute

- Calibration, instrument, and random uncertainty in the thermistor measurement.
- Calibration, instrument, and random uncertainty in the thermocouple measurement.

Example: T-couple Measurement with Scanning Multimeter (5)

Suppose we have the following data taken when the system is assumed to be at steady state

$$\bar{E} = -1.15 \times 10^{-5}\,\text{V}$$
average of 30 readings of the thermocouple referenced to the zone box

$$\sigma_E = 2.97 \times 10^{-6}\,\text{V}$$
standard deviation of 30 thermocouple readings

$$\bar{R} = 10815.3\,\Omega$$
average of 30 readings of the thermistor in the zone box

$$\sigma_R = 4.0\,\Omega$$
standard deviation of 30 thermistor readings

Example: T-couple Measurement with Scanning Multimeter (6)

Assume that the instrument error is zero (negligible) for both the DC voltage reading and the resistance reading. This is reasonable for the Agilent multimeter. Recall that the resistance reading is a 4-wire measurement that eliminates the resistance of the lead wire.

Example: T-couple Measurement with Scanning Multimeter (7)

The calibration equation for the temperature reading is of the form

$$T = f(E, R)$$

where E is the EMF of the thermocouple, and R is the resistance of the thermistor.

Using the sequential perturbation technique, and assuming $u_{T_{\mathrm{inst}}} = 0$ for both the voltage and resistance measurements, the uncertainty in the temperature reading is

$$u_T = \sqrt{u_{T_{\mathrm{rand}, E}}^2 + u_{T_{\mathrm{rand}, R}}^2 + u_{T_{\mathrm{cal}}}^2}$$

where

$$u_{T_{\mathrm{rand}, E}} = f(E + 2\sigma_E, R) - f(E, R)$$

$$u_{T_{\mathrm{rand}, R}} = f(E, R + 2\sigma_R) - f(E, R)$$

The factor of 2 in the perturbation gives an uncertainty estimate with 95 percent confidence intervals (20:1 odds).

Example: T-couple Measurement with Scanning Multimeter (8)

In MATLAB the calculations are

```
>> E = -1.15e-5;  sigE = 2.97e-6;  R = 10815.3;  sigR = 4;
>> Tnom = Ttemp(E,ThermistorT(R))
Tnom =
   22.7939

>> uTrandE = Ttemp(E+2*sigE,ThermistorT(R)) - Tnom
uTrandE =
    0.1465

>> uTrandR = Ttemp(E,ThermistorT(R+2*sigR)) - Tnom
uTrandR =
   -0.0181

>> uTcal = 0.1;
>> uT = sqrt(uTrandE^2 + uTrandR^2 + uTcal^2)
uT =
    0.1783
```

Example: T-couple Measurement with Scanning Multimeter (9)

In MATLAB the calculations give $u_T = 0.178\,°\text{C}$.

From this calculation we conclude that the uncertainty in the temperature measurement is $\pm 0.2\,°\text{C}$ at 20:1 odds.

Note that in the preceding calculations we have neglected errors in the curve fits to the calibration data.

Summary

1. Identify contributions to measurement errors
   - Bias errors: correct with centering models
   - Uncontrollable errors: report uncertainty

2. Estimate uncertainty in measured values due to
   - Instrumentation errors
   - Calibration errors
   - Random errors – use statistical sampling

3. Compute uncertainty in the result with sequential perturbation method
   - Develop data reduction formulas *including centering corrections*
   - Uncertainty in *raw measurements* (e.g. EMF) are due to random fluctuations and possibly instrumentation uncertainties
   - Uncertainty in *converted values* (e.g. temperature) are due to uncertainties in the raw measurements and due to uncertainties in calibration formulas.
   - The sequential perturbation method accounts for the propagation of uncertainties in raw measurements and converted values. This includes uncertainties introduced by the measured values used in any centering corrections.

References

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