Fast Minimization of Mixed-Polarity AND/XOR Canonical Networks*

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Abstract

*A quasi-minimal algorithm for Canonical Restricted Mixed Polarity (*CRMP*) AND/XOR forms is presented. These forms, which include the Consistent and Inconsistent Generalized Reed-Muller forms, are both very easily testable and, on average, have smaller number of terms than* SOP *expressions. The set of test vectors to detect stuck-at and bridging faults of a function realized in* CRMP *forms, similar to that of Consistent Generalized Reed-Muller (*CGRM*) forms, is independent of the function. This test set can be of order* $(n + 4)r$*, where* n *is the number of variables in the function and* r *is the the number of component* CGRM*s in the* CRMP*. The experimental results confirm the compactness of* CRMP*s as compared to* SOP *expressions.*

1 Introduction

With certain advantages of AND/XOR logic and the new technologies which make their use more practical, there is a need for new XOR-based logic synthesis methods. In this paper a quasi-minimal synthesis program for mixed-polarity AND/XOR canonical networks is introduced. These networks on the average have smaller number of product terms than the Sum of Products realizations and the number of tests required for them is close to those for the fixed-polarity AND/XOR canonical networks, which are the most easily testable of all general purpose networks.

It has long been the experience of logic designers that AND/XOR forms in certain cases are more economical than the conventional realizations using AND/OR/NAND/NOR forms. While this has been theoretically shown for the case of AND/XOR PLAs and AND/OR PLAs [14], it has also been confirmed practically on many examples, especially in arithmetic and telecommunication circuits [7, 8, 13]. In addition, this logic is highly testable and the test vector for detection of stuck-at-faults and bridging faults can be made independent of the function itself [11, 12].

While the slow speed of the XOR gate has been a major obstacle to the use of this logic, new technologies provide the means for its increased use. The AND/XOR PLAs have been extensively studied and the layout generators for them have been presented in [6]. Several families of new PLD devices have been recently marketed: Table Look-up based (Xilinx) and multiplexor-based (Actel 1020) Field Programmable Gate Arrays, folded NAND devices (Signetics LHS501), and XOR PLDs. Such devices either directly include XOR gates (LHS501) or allow to realize them in "universal modules". Since the five input XOR gate in Xilinx has the same speed and cost as, for instance, a five input OR gate, XOR gates can be used on equal terms with AND and OR gates.

Of particular interest are the CLi6006 FPGAs from Concurrent Logic, Inc. [3]. For these devices, each logic function must be constructed from two-input gates: XORs, ANDs and NANDs, and inverters. The multi-level XOR based logic is a prime candidate for mapping to such devices.

There are two basic approaches to synthesis using XOR gates: Exclusive Sums of Products (*ESOPs*), and Consistent Generalized Reed-Muller (*CGRM*) forms. The problem of finding the minimal *ESOPs* of a Boolean function [1, 8, 15, 16, 17] is a classical one in logic synthesis theory but exact solutions to it have been proposed for only small functions. The situation is better in the case of the CGRM canonical form of optimal polarity for which efficient exact and approximate algorithms have been recently created [13]. However, for some functions, the minimum CGRM can be much worse than the *ESOP*.

In [4], a class of canonical forms, Canonical Restricted Mixed-Polarity(*CRMP*), also known as Generalized Reed-Muller(*GRM*) forms, was studied. It was shown that in the worst case, the number of terms for *CRMPs* is the same as *ESOPs*, which is 3/4 of that for the *SOPs*.

In this paper, following a discussion of properties and testability of *CRMP* forms, a quasi-minimization scheme for these forms will be presented. The scheme is next tested on a number of functions and the results are provided.

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2 Background

The *CRMP* forms are a class of AND/XOR canonical forms [5]. The *CRMPs* are of the form:

$$f(x_1, x_2, \ldots, x_n) = \bigoplus_{i=0}^{2^n-1} a_i \mu_i \quad (1)$$

where $a_i \in \{0, 1\}$ and the term $\mu_i = \dot{x}_n^{e_n} \dot{x}_{n-1}^{e_{n-1}} \cdots \dot{x}_2^{e_2} \dot{x}_1^{e_1} = \prod_{j=1}^{n} \dot{x}_j^{e_j}$ where $e_j \in \{0, 1\}$ such that $e_n e_{n-1} \ldots e_2 e_1$ is a binary number which equals i. Moreover $\dot{x}_i^0 = 1$ and $\dot{x}_i^1 = \dot{x}_i$, where $\dot{x}_i = x_i$ or $\bar{x}_i$. $\oplus$ denotes summation over GF(2), the Galois field of two elements.

If there is a restriction on each variable to have only positive or negative polarity in all the terms, the form will be a Consistent Generalized Reed-Muller(*CGRM*) form. If the variables are restricted to take only positive polarities, the form will be that of Reed-Muller Canonical(*RMC*) form. The *CRMPs* and all the above forms are canonical [5]. A superset of all AND/XOR forms is that of *ESOP* expressions, which are not necessarily canonical and are the most economical of AND/XOR representations.

Example 1 $1 \oplus x_1 \oplus \bar{x}_2 \oplus \bar{x}_1 x_2$ *is a* CRMP *form, because there exists only one term for each subset of variables. It is not a* CGRM *form because* x_1 *appears both in a negative and a positive polarity.* $x_1\bar{x}_2 \oplus \bar{x}_1 x_2 \oplus x_2 \oplus x_1 x_2$ *is an* ESOP *which is not in a* CRMP *form because the term* $x_1 x_2$ *occurs more than once (in different polarities).* $\bar{x}_2 \oplus \bar{x}_1 \bar{x}_2$ *is a* CGRM *form, since both variables occur in constant polarities in the entire form. This is not a* RMC *form because variables are not positive. It is called a negative Reed-Muller form.*

For a function of n variables, there are 2^n possible *CGRM* representations of the function and $2^{n2^{n-1}}$ possible *CRMP* representations. As there are more *CRMP* forms for a given function, a minimal solution close to *ESOPs* is more likely than for *CGRMs*.

Definition 1 *The* Boolean difference *of function f with respect to variable x_i is denoted by f_{x_i} and is defined as[5]:*

$$f_{x_i} = f(x_1, \ldots, x_i, \ldots, x_n) \oplus f(x_1, \ldots, \bar{x}_i, \ldots, x_n). \quad (2)$$

Definition 2 *The* Boolean difference *of function f with respect to term $t = \dot{x}_i \dot{x}_j \ldots \dot{x}_r$ is denoted by f_t and is defined as:*

$$f_t = (\ldots(f_{\dot{x}_i})_{\dot{x}_j} \ldots)_{\dot{x}_r}. \quad (3)$$

Definition 3 *Let t be a term. The* term set S(t) *of t is $S(t) = \{x_i \mid \dot{x}_i$ appears in $t\}$.*

Definition 4 *Term t is a* prime term *with respect to function f iff $f_t \equiv 1$, where $\equiv$ stands for identical equality.*

Theorem 1 *Term t is a prime term with respect to function f iff in any* CRMP *form of f there exists exactly one term t' such that $S(t) = S(t')$ and there exists no term t'' such that $S(t) \subset S(t'')$.*

The implication of the above theorem is that every function in a *CRMP* form, including the minimal *CRMP* form, has at least one prime term. In addition, for all existing terms $\underline{t}$ of a *CRMP* form of f, there is a prime term t such that $S(\underline{t}) \subseteq S(t)$.

Definition 5 *Term t is a* nonexisting term *with respect to function f iff $f_t \equiv 0$.*

Theorem 2 *Term t is a nonexisting term with respect to function f iff in any* CRMP *form of f there is no term t' such that $S(t) \subset S(t')$.*

3 Testability of CRMP Forms

Several "design for testability" and test generation methods for *RMC* form have been introduced [2, 10, 11, 12]. Those tests are universal and the number of test vectors are the smallest possible for all known types of universal circuit structures.

It can easily be shown that any universal test set generated for detection of single stuck-at-faults of an *RMC* form can be modified for any *CGRM* form by just inverting the test bits for those variables which are of negative polarity in the *CGRM*. Similar results hold true for multiple stuck-at and bridging faults [2, 10, 11]. A *CGRM* of arbitrary polarity is then as easily testable as the *RMC* form of a function.

Since, in general, a *CRMP* is an exclusive sum of its component *CGRMs* of various polarities, the test set for the *CRMP* can be created as a composition of the test sets of its component *CGRMs*[10]. With separately observable *CGRM* components, tests are generated using any of the known methods for each component *CGRM* separately. The universal test-generator circuit can just use a modification for the *CRMP* network. As for a function with n variables, $n+4$ tests can detect the stuck-at and bridging faults of each component *CGRM*, $r(n + 4)$ tests can then detect all the faults in *CRMP* network independent of the function, where r is the number of the component *CGRMs*.

4 Quasi-Minimal CRMP Form

In this section, a quasi-minimal method for generation of a *CRMP* form of a function with the least number of terms will be presented. The method is based on the notion of prime terms and relies on a fast *CGRM* minimizer [13]. In the following, certain results about prime terms will be given without proofs (for details see [4]). A quasi-minimal algorithm based on these results is then given at the end of the section.

Definition 6 *Term t_1 is a* proper subcombination *of term t_2, iff $S(t_1) \subset S(t_2)$.*

As an example, in function $f = x_1 x_2 x_3 \bar{x}_4 \oplus \bar{x}_1 x_3$, the term $\bar{x}_1 x_3$ is a proper subcombination of the term $x_1 x_2 x_3 \bar{x}_4$.

Theorem 3 *All terms of a Boolean function, f, of n variables given in a* CRMP *form which are not subcombinations of other terms in the same* CRMP *form will exist in any* CRMP *form of f (possibly with a different polarity of variables).*

From Theorem 3, it results that the prime terms will exist in the minimal *CRMP* form too. Also, all existing terms in any *CRMP* form of a Boolean function, f, of n variables are subcombinations of prime terms. In addition, for a given Boolean function, f, of n variables, the prime terms are entirely determined by f and they do not depend on the *CRMP* form from which they are determined.

Theorem 4 *If term t of a Boolean function, f, of n variables given in a* CRMP *form does not exist and is not a subcombination of the prime terms of f, then in no other* CRMP *form of f can there be a term t′ such that* $S(t) = S(t')$.

Example 2 *Based on above properties, in the following, some functions are given with their prime terms underlined and their nonexisting terms listed.*

1. $1 \oplus x_1 \oplus x_2 \oplus \underline{x_1 x_2 x_3}$. *Nonexisting terms: none.*
2. $1 \oplus x_1 \oplus \underline{x_1 x_2} \oplus \underline{x_1 x_3} \oplus \underline{x_2 x_3}$. *Nonexisting terms:* $x_1 x_2 x_3$.
3. $1 \oplus \underline{x_1} \oplus x_2 \oplus \underline{x_2 x_3}$. *Nonexisting terms:* $x_1 x_2, x_1 x_3, x_1 x_2 x_3$.
4. $x_1 \oplus x_2 \oplus \underline{x_3} \oplus x_4 \oplus \underline{x_1 x_2 x_4}$. *Nonexisting terms:* $x_1 x_3$, $x_2 x_3, x_3 x_4, x_1 x_2 x_3, x_1 x_3 x_4, x_2 x_3 x_4$, $x_1 x_2 x_3 x_4$.
5. $x_1 \oplus x_2 x_3 \oplus \underline{x_1 x_2 x_3 x_4}$. *Nonexisting terms: none.*
6. $\underline{x_1} \oplus \underline{x_2 x_3} \oplus \underline{x_2 x_4} \oplus \underline{x_3 x_4}$. *Nonexisting terms:* $x_1 x_2$, $x_1 x_3$, $x_1 x_4, x_1 x_2 x_3, x_1 x_2 x_4$, $x_1 x_3 x_4$, $x_2 x_3 x_4$, $x_1 x_2 x_3 x_4$.

If there exist only prime terms in the expression, then this expression is a both term-wise and literal-wise minimal *CRMP* form. If one can merge other terms into the prime terms so that the resultant form has the same number of terms as the prime terms, the resultant form is also an exact minimal one.

Based on the above theorems and the properties indicated, a depth-first search algorithm has been devised. This algorithm, called CANNES (CANonic Nor Exor Synthesizer), is based on the fact that all prime terms are entirely determined from the Boolean function, f, and they do not depend on any *CRMP* form and that all existing terms in a *CRMP* form of f are subcombinations of prime terms. In this sense, CANNES is an algorithm which generates the minimal *CGRM* form for the prime terms and their subcombinations.

The simplified recursive minimization procedure of CANNES is as follows:

CANNES
{
  for (each prime term *of the* **List** *)* {
    // find the subset for the prime term
    subset = subset_of(prime term);
    // find the minimal CGRM *form of the subset*
    minsubset = minimal_CGRM(subset);
    // compare number of terms
    if (| *minsubset* | < | *subset* | *)*
    {
      NewList = **List**;
      replace subset *in* **NewList** *by* minsubset;
      minimize(**NewList** *)*;
  } } }

List	- complete list of terms in the function,
NewList	- starting **List** of next recursion,
prime term	- prime term of the **List**,
subset	- subset of terms for a *prime term*,
minsubset	- minimal *CGRM* form of the *subset*.

CANNES-2 uses a heuristic *CGRM* minimizer [13] to find minimal *CGRM* forms for subsets of variables. For the exact *CGRM* minimizer, while our method requires searching all polarities of a *CGRM*, and even several times during the *CRMP* minimization, it is usually done on a subfunction of the initial function. Only in the worst case of a single prime term, the polarities of all input variables are searched. Concluding, with an amount of search that is comparable to that of a *CGRM*, we are able to find a form that is not worse than the *CGRM*.

5 Experimental Results

CANNES-2 was tested on 100 single output functions generated from the MCNC benchmarks. Table 1 shows the number of terms for ESPRESSO, CANNES-2, and EXORCISM [17], an *ESOP* minimizer, for some of these functions. In this table, n stands for the number of variables in the functions.

For the functions tested, the compactness of AND/XOR forms is confirmed. While for the 100 functions overall, ESPRESSO resulted in 1001 terms, CANNES-2 gave 845 and EXORCISM 652. For 40 percent of the functions, CANNES-2 gave better results than ESPRESSO while for 30 percent, ESPRESSO gave fewer terms. For the rest, they both gave the same number of terms. Some of the examples of these cases are shown in Table 1. Moreover, for all small functions that can be verified (such as all single output functions of three and many functions of four variables), the algorithm produced the exact *CRMP* solutions. Whether the algorithm always gives the exact solution needs to be studied further.

6 Conclusions

A quasi-minimal *CRMP* form has been introduced. We have shown that testability of *CRMP*s is close to

Name	n	ESPRESSO	CANNES-2	EXORCISM
5xp11	7	7	9	6
9sym	9	85	131	51
majority	5	5	6	5
bw7	5	6	5	5
con12	7	5	4	4
duke8	22	5	4	5
f51m4	8	10	6	5
rd532	5	16	5	5
rd732	5	64	7	7
rd842	8	128	8	8
mis70	5	6	7	5
vg28	25	5	9	7
z42	7	28	9	9

Table 1: Two Level AND/OR Compared to Two Level *CRMP* and *ESOP*

*RMC*s and their costs close to *ESOP*s. Such forms should be then studied in more detail, especially with respect to the "design for testability".

The concept of prime terms allows to *decompose* a function to disjoint subsets of input variables for which minimization can be done separately. This allows for a parallel approach to the minimization problem.

Moreover, these results are important with respect to the logic design methods for FPGAs and other programmable devices and technologies. Our results show that a *CRMP* form can not only be much more easily testable, but also smaller than the two-level inclusive form. Since the cost and speed of XOR gate and OR gate in these technologies are the same, there is no reason to always use PLA minimizers such as ESPRESSO for them. The AND/OR forms can be used for certain functions which do not have a compact AND/XOR realization. The best design should be selected from the factored AND/OR and AND/XOR solutions. Particularly, the same approach to factorization of AND/OR circuits can be applied to the factorization of AND/XOR circuits [10], leading to the same kinds of advantages. Moreover, Multi-level AND/OR/XOR realizations of functions can be formulated and respective synthesis methods created [9, 10].

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