Fast Synthesis for Ternary Reed–Muller Expansion

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Abstract

Since the multiple valued circuits based upon the Reed-Muller expansion have the advantages in testability and diagnosis[1–3], much attention has been paid to the investigation on the multiple valued expansion in recent years[4–10]. Many authors have presented various algorithms of calculating the coefficients of RM expansions under fixed polarities. A polarity maxtrix transform method has been presented in Ref.[4,5], but its computational cost is very high. A fast flow graph algorithm was developed in Ref.[6], which reduces the computational cost in comparison with the polarity matrix transform method but does not improve much. A step by step flow graph algorithm with the polarities in Gray code order is proposed in Ref.[7,8], whose computational cost is the lowest of all existing algorithms, but it can not be implemented in fast parallel computation due to the step by step serial computation. The RM coefficients mapping simplification in Ref.[9,10] is direct and easy but is not suitable to RM functions with multiple variables.

In this paper, we derive out the direct algorithm of caculating RM coefficients under each fixed polarity. This algorithm has not only a simple procedure but also much lower computational cost than the step-by-step flow graph algorithm with the polarities in Gray code order in Ref.[7]. Therefore, it can be implemented in fast parallel computation.

I. Direct Algorithm for Polarity Matrix

The canonical RM expansion for an any n-variable ternary function may be written as

$$f(x_1,x_2,\ldots,x_n) = b_0 x_1^0 x_2^0 \ldots x_n^0 \oplus b_1 x_1^0 x_2^0 \ldots x_n^1 \oplus \cdots \oplus b_j x_1^{e_1} x_2^{e_2} \ldots x_n^{e_n} \oplus \ldots \oplus b_{3^n-1} x_1^2 x_2^2 \ldots x_n^2 \tag{1}$$

where, $x_i^0 = 1, x_i^1 = x_i, x_i^2 = x_i \cdot x_i$ $(i = 1,2,\ldots,n)$. $e_i, b_j \in \{0,1,2\}$, $(j = 0,1,2,\ldots,3^n-1)$, and the j is the decimal expression of the ternary number $e_1e_2,\ldots,e_n$, that is, $<(j)>_{10}=<(e_1e_2\ldots e_n)>_3$. $B = (b_0,b_1,\ldots,b_n)$ is called the cofficient vector under the zero-polarity for the function f.

Let $g(\dot{x}_1,\dot{x}_2,\ldots,\dot{x}_n) = f(x_1,x_2,\ldots,x_n), \dot{x}_i = \dot{x}_i \oplus \alpha_i$, $\alpha_i \in \{0,1,2\}$ and let k be the decimal expression of the ternary number $\alpha_1\alpha_2\ldots\alpha_n$, that is, $<k>_{10}=<\alpha_1\alpha_2\ldots\alpha_n>_3, k = 0,1,2,\ldots,3^n-1$. Obviously, $g(\dot{x}_1,\ldots,\dot{x}_n)$ has 3^n different forms while k takes the different value.

Definition 1. The polarity matrix $M(f)$ of $f(x_1,x_2,\ldots,x_n)$ is a $3^n \times 3^n$ matrix, whose $(k+1)$-th row represents the coefficient vector $B^{(k)}$ of the RM expansion for f with the polarity k, where $B^{(0)}$ is the vector under the zero-polarity. Obviously, $B^{(0)} = B$ and $M(f) = (B^{(0)},\ldots,B^{(3^n-1)})^T$.

Definition 2. The optimum polarity of $f(x_1,x_2,\ldots,x_n)$ denotes that under the optimum polarity k the coefficient vector $B^{(k)}$ has the minimum non-zero elements.

Let the RM expansion for an $(n+1)$-variable ternary function $f(x_0, x_1, \ldots, x_n)$ be:

$$\begin{aligned} f(x_1,x_2,\ldots,x_n) =& (b_0 \oplus b_1 x_n \oplus \ldots \oplus b_{3^n-1} \\ & x_1^2 x_2^2 \ldots x_n^2) \oplus (b_{3^n} \oplus b_{3^n+1} x_n \oplus \ldots \\ & \oplus b_{2\cdot 3^n-1} x_1^2 x_2^2 \ldots x_n^2) x_0 \oplus (b_{2\cdot 3^n} \\ & \oplus b_{2\cdot 3^n+1} \oplus \ldots \oplus b_{3^{n+1}-1} x_1^2 x_2^2 \ldots x_n^2) x_0^2 \\ =& f' \oplus f'' x_0 \oplus f''' x_0^2 \end{aligned} \tag{2}$$

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Let the zero-polarity coefficient vectors corresponding to f', f'', f''' are B', B'', B''', and thus we have:

$$B = B^{(0)} = (B', B'', B''')$$

Where, $B' = (b_0, \ldots, b_{3^n-1})$, $B'' = (b_{3^n}, \ldots, b_{2\cdot 3^n-1})$, and $B''' = (b_{2\cdot 3^n}, \ldots, b_{3^{n+1}-1}$.

Definition 3. Define $3^{n+1} \cdot 3^{n+1}$ recurrence matrix $N(B)$

$$N(B) = \begin{pmatrix} N(B') & N(B'') & N(B''') \\ N(B' \oplus 2B'' \oplus B''') & N(B'' \oplus B''') & N(B''') \\ N(B' \oplus B'' \oplus B''') & N(B'' \oplus 2B''') & N(B''') \end{pmatrix}$$

Where, $N(b_j) = b_j$ [3]

Theorem 1. $N(B) = M(f)$. (4)

Proof: By using the inductive method, when $i = 1$, We have

$$\begin{aligned} f(x) = g(\dot{x}) &= b_0^k \oplus b_1^k \dot{x} \oplus b_2^k \dot{x}^2 = b_0 \oplus b_1 x \oplus b_2 x^2 \\ &= b_0^{(1)} \oplus b_1^{(1)} \tilde{x} \oplus b_2^{(1)} \tilde{x}^2 = b_0^{(2)} \oplus b_1^{(2)} \tilde{\tilde{x}} \oplus b_2^{(2)} \tilde{\tilde{x}}^2 \end{aligned} \quad (5)$$

where $\tilde{x} = x \oplus 1, \tilde{\tilde{x}} = x \oplus 1$. Then, substitute $\tilde{x} \oplus 2, \tilde{\tilde{x}} \oplus 2$ for x in Equ.(5). By comparing their coefficients, we obtain

$$(b_0^{(1)}, b_1^{(1)}, b_2^{(1)}) = (b_0 \oplus 2b_1 \oplus b_2, b_1 \oplus b_2, b_2),$$

and

$$(b_0^{(2)}, b_1^{(2)}, b_2^{(2)}) = (b_0 \oplus b_1 \oplus b_2, b_1 \oplus 2b_2, b_2),$$

and thus we have

$$M(f) = \begin{pmatrix} b_0 & b_1 & b_2 \\ b_0 \oplus 2b_1 \oplus b_2 & b_1 \oplus b_2 & b_2 \\ b_0 \ominus b_1 \oplus b_2 & b_1 \oplus b_2 & b_2 \end{pmatrix} = N(B)$$

Suppose that the Equ.(4) is right when $i = n$. Thus, when $i = n+1$ we can derive out

$$\begin{aligned} f(x_0, x_1, \ldots, x_n) = g(\dot{x}^0, \dot{x}^1, \ldots, \dot{x}^n) &= f^{(k)'} \oplus f^{(k)''} \dot{x}_0 \oplus f^{(k)'''} \dot{x}_0^2 \\ &= f' \oplus f'' x_0 \oplus f''' x_0^2 = f^{(1)'} \oplus f^{(1)''} \tilde{x}_0 \oplus f^{(1)'''} \tilde{x}_0^2 \\ &= f^{(2)'} \oplus f^{(2)''} \tilde{\tilde{x}}_0 \oplus f^{(2)'''} \tilde{\tilde{x}}_0^2 \end{aligned} \quad (6)$$

from the Equ.(2).

Let the polarity matrixes corresponding to f' , f'' and f' are $M(f')$, $M(f'')$ and $M(f''')$, respectively. Consider 3^{n+1} polarities of $f(x_0, x_1, \ldots, x_n)$, $< k >_{10} = < \alpha_0, \alpha_1, \ldots, \alpha_n >_3$, $\dot{x}_i =^{\Delta} x_i \oplus \alpha_i$, $i = 0, 1, \ldots, n$. $\alpha_i \in \{0, 1, 2\}$, and we find:

(1) When $0 \leq k \leq 3^n - 1, \dot{x}_0 = x_0$, whose corresponding polarity matrix is $(M(f'), M(f''), M(f'''))$, that is, $(N(B'), N(B''), N(B'''))$.

(2) When $3^n \leq k \leq 2\cdot 3^n - 1, \dot{x}_0 = x_0 \oplus 1$, from Equ.(6) we can obtain $f^{(1)'} = f' \oplus 2f'' \oplus f''', f^{(1)''} = f'' \oplus f''', f^{(1)'''} = f'''$, whose corresponding matrix is $(M(f' \oplus 2f'' \oplus f'''), M(f'' \oplus f'''), M(f'''))$.

Since M, N are both the linear transform, the polarity matrix can be rewritten as $(N(B' \oplus 2B'' \oplus B'''), N(B'' \oplus B'''), N(B'''))$.

(3) When $2\cdot 3^n \leq k \leq 3^{n+1} - 1, \dot{x}_0 = x_0 \oplus 1$, from Equ.(6) we can obtain $f^{(2)'} = f' \oplus f'' \oplus f''', f^{(2)''} = f'' \oplus 2f''', f^{(2)'''} = f'''$, whose corresponding polarity matrix is $(M(f' \oplus f'' \oplus f'''), M(f'' \oplus 2f'''), M(f''')) = (N(B' \oplus B'' \oplus B'''), N(B'' \oplus 2B'''), N(B'''))$.

Thus, we prove that Equ.(4) is right when $i = n+1$. From the inductive method we conclude that Equ.(4) is right for any arbitrary positive integer i.

Example. Let $f(x_1, x_2) = x_2 \oplus x_2^2 \oplus x_1 \oplus 2x_1x_2 \oplus x_1x_2^2 \oplus x_1^2 \oplus 2x_1^2x_2 \oplus x_1^2x_2^2$.

Evaluate the polarity matrix of $f(x_1, x_2)$ solution. The zero polarity RM coefficient vector is $B = B^{(0)} = (011121121)$. From theorem 1 we have its polarity matrix

$$M(f) = N(B) = \begin{pmatrix} N(011) & N(121) & N(121) \\ N(011) & N(102) & N(121) \\ N(220) & N(000) & N(121) \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 1 & 1 & 2 & 1 & 1 & 2 & 1 \\ 0 & 2 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 2 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 2 & 1 & 1 & 2 & 1 \\ 0 & 2 & 1 & 0 & 0 & 2 & 0 & 0 & 1 \\ 2 & 0 & 1 & 2 & 2 & 2 & 1 & 1 & 1 \\ 2 & 2 & 0 & 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 2 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

Green presented a fast step-by-step flow graph algorithm with the polarities in Gray code order to evaluate the coefficients of the RM expansion over GF(3), whose computation complexity is the most advantageous of all exsiting algorithms. Let A_n be the number of additions, and M_n the number of multiplications. From Ref.[7] we have

$$A_n = 3^n \times (3^n - 1) \qquad M_n = A_n / 3$$

Compare the algorithm in this paper with the algorithm in Ref.[7]. If our algorithm is realized in the computation order as follows,

$$B'' \xrightarrow{\oplus B'''} B'' \oplus B''' \xrightarrow{\oplus B'''} B'' \oplus 2B'''$$

$$B'' \oplus B''' \xrightarrow{\ominus B'} B' \oplus B'' \oplus B''' \xrightarrow{\oplus B''} B' \oplus 2B'' \oplus B'''$$

obviously, only four additions are required, and no multiplication is included.

Consider the computation of $N(B)$ from B. From Equ.(3) at the first level $(i = 1)$, $4 \times 3^{n-1}$ additions are required, and at the second level $(i = 2)$, only 7 different sub-matrixes are needed to compute, so $7 \times 4 \times 3^{n-1}$ additions are required. On the analogy of this, at the i-th level $7^{i-1} \times 4 \times 3^{n-i}$ additions are required. Thus, we obtain the total number of additions: $A_n = \sum_{i=1}^{n} 7^{i-1} \cdot 4 \cdot 3^{n-i} = 7^n - 3^n$, and the total number of multiplications $M_n = 0$. Compare with the computation complexity for the above two algorithms by listing Table 1.

Table 1. Comparison of two algorithmic complexities

	Green Fast Algorithm		Our Algorithm	
n	A_n	M_n	A_n	M_n
1	6	2	4	0
2	72	24	40	0
3	702	234	316	0
4	6,480	2,160	2,320	0
5	58,806	19,602	16,564	0
6	530,712	176,904	116,920	0
7	4,780,782	1,593,594	821,356	0
8	43,040,160	14,346,720	5,758,240	0

Obviously from Table 1, our results are much more advantageous to Green's.

IV. Conclusion

In conclusion, our proposed algorithm of evaluating ternary RM expansion coefficients under fixed polarities has the following major advantages:

1. Its computation is much less than that of Green's flow graph with polarities in Gray code order, especially when the number of variables is large.

2. When the number of variables increases, the computation procedure of our algorithm is a recurrence one, which can be easily implemented with computer program in parallel computation.[11]

Finally, it should be pointed out that although only ternary RM function are disscussed in this paper, its thoughts are easily extended to the multiple-valued RM functions with higher radix.

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