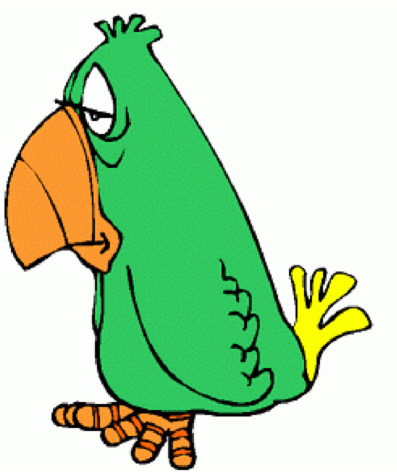
In This Talk

“Zee” = “Zee”

Advanced Topics In Quantum Error Correction*

Parrots, Reptiles, and Donuts



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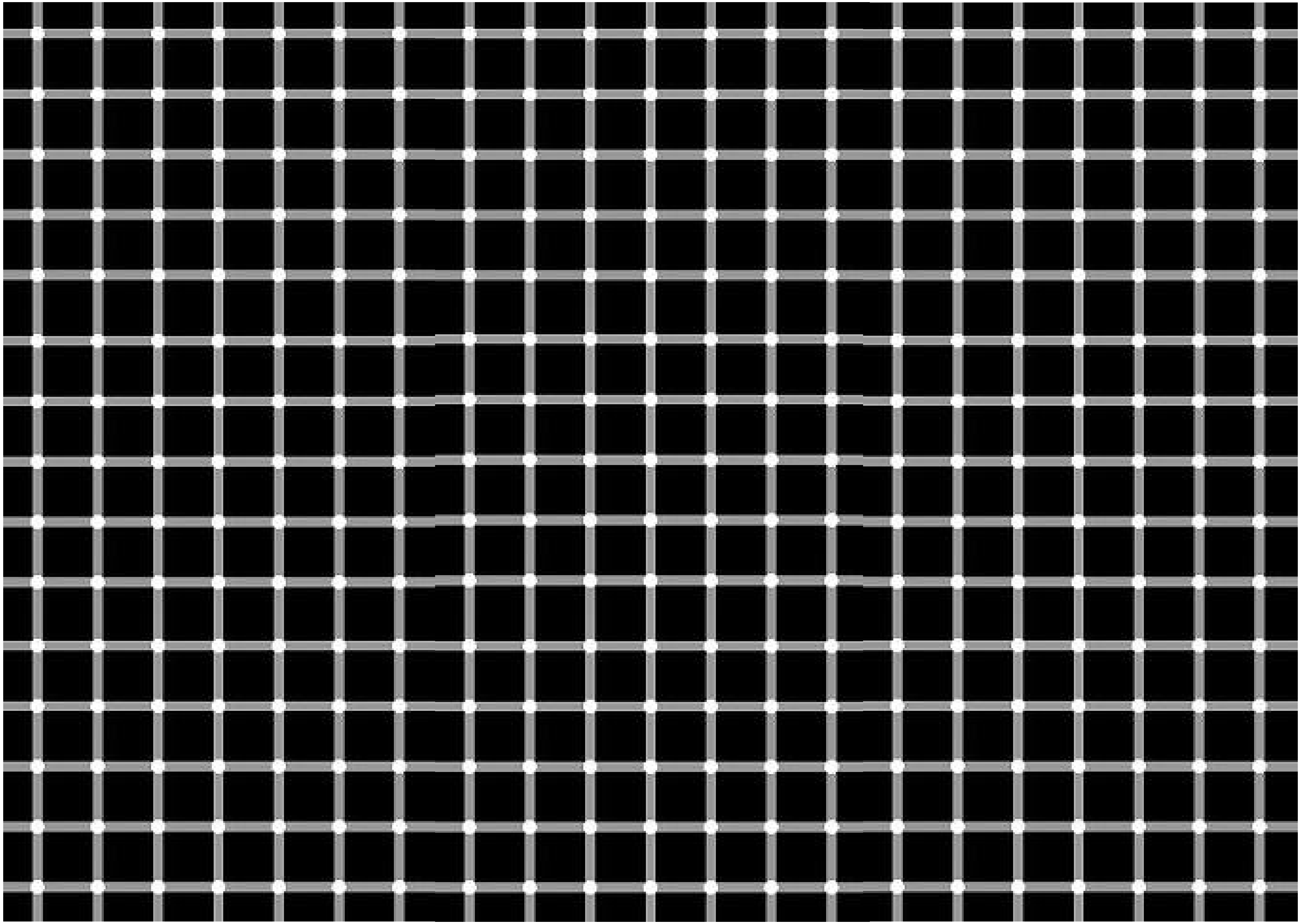
WARNING

This Talk Under Constant Acceleration

DB and sQuInT assume no responsibility for injuries sustained while zoning out.

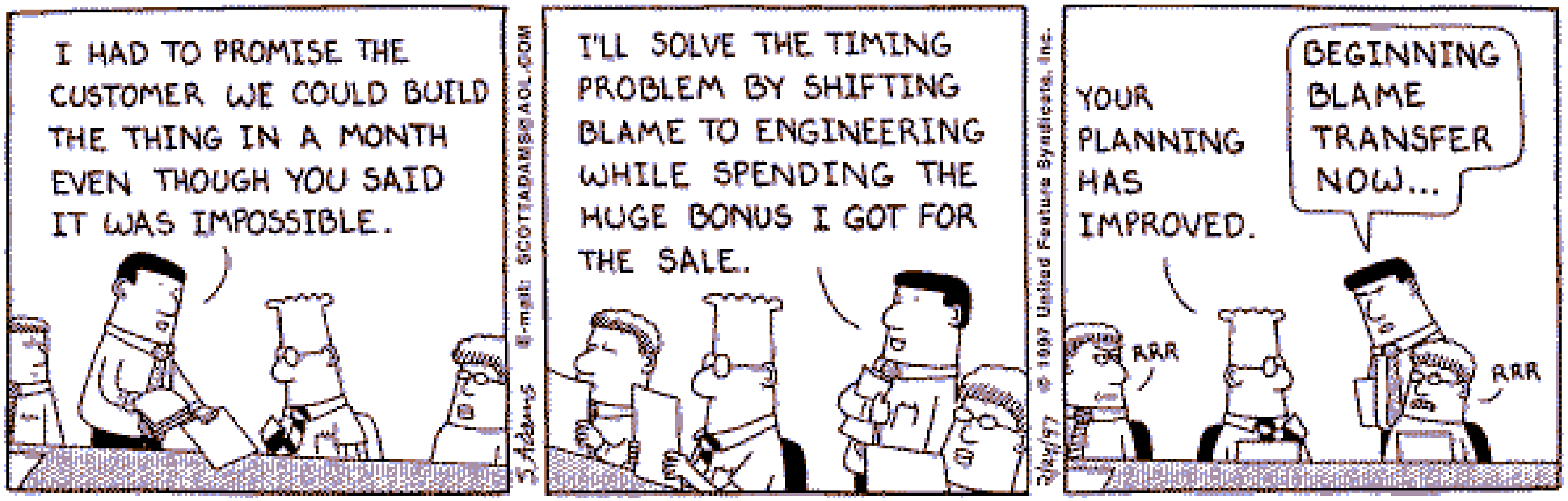


Schedule

- **The Pauli Group, Stabilizer Codes, and All That**

- **Topological Quantum Codes**

If I misunderestimate TIME, and your stomach REVOLTS, please have your stomach SPEAK UP and tell me to shut my trap.



Quick Review

"The past exists only as recorded in the present."

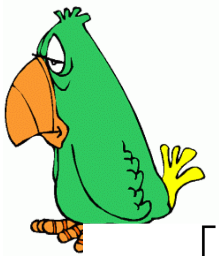
- Operator Sum Representation (OSR) of open system evolution:

$$\boldsymbol{\rho} \rightarrow \sum_k \mathbf{A}_k \boldsymbol{\rho} \mathbf{A}_k^\dagger, \quad \sum_k \mathbf{A}_k^\dagger \mathbf{A}_k = \mathbf{I}$$

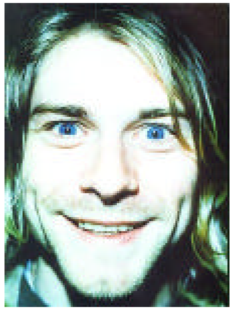
- Bit flip code overcomes no-cloning.
- Phase errors corrected by bit flip code in different basis.
- Shor code corrects single qubit errors.
- Quantum Error Correcting Code iff condition

$$\langle i | \mathbf{E}_a^\dagger \mathbf{E}_b | j \rangle = C_{ab} \delta_{ij}$$

- Errors can be "digitized".
- Argued for independent error model.



Pauli Want a Cracker?



$$\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \mathbf{X} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \mathbf{Y} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad \mathbf{Z} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\mathbf{XY} = i\mathbf{Z} \quad \mathbf{YX} = -i\mathbf{Z} \qquad \{\mathbf{X},\mathbf{Y}\} = \{\mathbf{X},\mathbf{Z}\} = \{\mathbf{Y},\mathbf{Z}\} = 0$$

def: Pauli group on 1 qubit: $\mathcal{P}_1 = \{\pm\mathbf{I}, \pm i\mathbf{I}, \pm\mathbf{X}, \pm i\mathbf{X}, \pm\mathbf{Y}, \pm i\mathbf{Y}, \pm\mathbf{Z}, \pm i\mathbf{Z}\}$

notation

$$\mathbf{P}_i = \mathbf{I} \otimes \cdots \otimes \mathbf{I} \otimes \underbrace{\mathbf{P}}_{ith\ qubit} \otimes \mathbf{I} \otimes \cdots \otimes \mathbf{I}$$

def: Pauli operator on n qubits: $i^k \mathbf{P}_1 \mathbf{P}_2 \cdots \mathbf{P}_n$

$$\mathbf{P}_i \in \mathcal{P}_1,\ k = 0 \ldots 3$$

def: Pauli group on n qubits: $\mathcal{P}_n = \{\text{all Pauli operators on } n \text{ qubits}\}$

examples: $i\mathbf{X}_1\mathbf{Z}_2\mathbf{Y}_3 \in \mathcal{P}_3$, $\mathbf{X}_1\mathbf{Z}_3 \in \mathcal{P}_4$, CNOT $\notin \mathcal{P}_2$

Pauli Group Facts

Any elements $\mathbf{P},\mathbf{Q} \in \mathcal{P}_n$ either commute or anticommute, but not both

$$[\mathbf{P},\mathbf{Q}] = 0 \quad \text{or} \quad \{\mathbf{P},\mathbf{Q}\} = 0$$

$$\begin{aligned}\mathbf{PQ} &= \mathbf{P}_1\mathbf{P}_2\cdots\mathbf{P}_n\mathbf{Q}_1\mathbf{Q}_2\cdots\mathbf{Q}_n = (\mathbf{P}_1\mathbf{Q}_1)(\mathbf{P}_2\mathbf{Q}_2)\cdots(\mathbf{P}_n\mathbf{Q}_n)\\ &= (\pm\mathbf{Q}_1\mathbf{P}_1)(\pm\mathbf{Q}_2\mathbf{P}_2)\cdots(\pm\mathbf{Q}_n\mathbf{P}_n) = \pm\mathbf{QP}\end{aligned}$$

$\mathbf{X}_1\mathbf{X}_2\mathbf{I}_3\mathbf{Z}_4$

$\mathbf{X}_1\mathbf{Z}_2\mathbf{Z}_3\mathbf{Y}_4$ ⇒ $(\mathbf{X}_1\mathbf{X}_2\mathbf{I}_3\mathbf{Z}_4)(\mathbf{X}_1\mathbf{Z}_2\mathbf{Z}_3\mathbf{Y}_4) = (\mathbf{X}_1\mathbf{X}_2\mathbf{I}_3\mathbf{Z}_4)(\mathbf{X}_1\mathbf{Z}_2\mathbf{Z}_3\mathbf{Y}_4)$

$\downarrow \; \downarrow \; \downarrow \; \downarrow$

$s \; d \; t \; d$

$s =$ same, $t =$ one is trivial, $d =$ different (none trivial)

$$d \text{ is even} \Rightarrow \mathbf{PQ} = \mathbf{QP} \Rightarrow [\mathbf{P},\mathbf{Q}] = 0$$
$$d \text{ is odd} \Rightarrow \mathbf{PQ} = -\mathbf{QP} \Rightarrow \{\mathbf{P},\mathbf{Q}\} = 0$$

Pauli elements, nonidentity are traceless $\mathrm{Tr}\,[\mathbf{P}] = 0$, $\mathbf{P} \neq i^k\mathbf{I}$ and all elements square to $\pm$ identity $\mathbf{P}^2 = \pm\mathbf{I}$

Set Your Square Phasors to +1

Focus on the elements of $\mathtt{P}_n$ such that $\mathbf{P}^2 = \mathbf{I}$:

- Eigenvalues of such $\mathbf{P}$ are ± 1: $\mathbf{P}^2|\psi\rangle = \lambda^2|\psi\rangle = |\psi\rangle \rightarrow \lambda = \pm 1$.
- Degeneracies of $+1$ and -1 eigenvalues are 2^{n-1} for nonidentity $\mathbf{P}$:

$$\mathbf{P} = \begin{bmatrix} +1 & 0 & \cdots & & \cdots & 0 \\ 0 & \ddots & & & & \vdots \\ \vdots & & +1 & & & \\ & & & -1 & & \vdots \\ \vdots & & & & \ddots & 0 \\ 0 & \cdots & & \cdots & 0 & -1 \end{bmatrix} \quad \begin{matrix} 2^{n-1} \ +1\text{'s} \\ \\ 2^{n-1} \ -1\text{'s} \end{matrix}$$

The $\mathbf{P}$ divide the space in half depending on the ± 1 eigenvalue.

Abelian not Reptilian

A *Pauli Stabilizer Group* on n qubits is an Abelian Subgroup of the Pauli Group on n qubits whose elements all square to identity.

$$\mathbb{S} = \{\mathbf{S}_\alpha \in \mathbb{P} | \mathbf{S}_\alpha^2 = \mathbf{I} \text{ and } [\mathbf{S}_\alpha, \mathbf{S}_\beta] = 0, \forall \alpha, \beta\}$$

Example: $\mathbb{S}_{ex} = \{\mathbf{IIII}, \mathbf{XXXX}, \mathbf{YYYY}, \mathbf{ZZZZ}\}$

The *generators* of a Pauli Stabilizer Group $\mathbb{S}$ is the smallest set whose elements can be multiplied to obtain the entire group [not unique].

Example: $\mathbb{S}_{ex}$ is generated by $\{\mathbf{XXXX}, \mathbf{ZZZZ}\}$

$$\mathbb{S}_{ex} = \langle \mathbf{XXXX}, \mathbf{ZZZZ} \rangle$$

Since group is abelian and square to identity, every element can be expressed as

$$\mathbf{S}_1^{\alpha_1} \mathbf{S}_2^{\alpha_2} \cdots \mathbf{S}_l^{\alpha_l}$$

where $\mathbf{S}_i$ generate the group and $\alpha_i \in \{0, 1\}$.

Example: $\mathbf{S} = (\mathbf{XXXX})^{\alpha_1} (\mathbf{ZZZZ})^{\alpha_2}, \ \forall \ \mathbf{S} \in \mathbb{S}_{ex}$

Pauli Stabilizer Code

A Pauli Stabilizer Code S for a Pauli Stabilizer Group $\mathbb{S}$ is defined as the common $+1$ eigenvalue eigenspace of the elements of $\mathbb{S}$:

$$|\psi_C\rangle \in S \text{ iff } \mathbf{S}_\alpha|\psi_C\rangle = |\psi_C\rangle, \ \forall \mathbf{S}_\alpha \in \mathbb{S}$$

Since

$$\mathbf{S}(\alpha_1, \alpha_2, \ldots, \alpha_k) = \mathbf{S}_1^{\alpha_1}\mathbf{S}_2^{\alpha_2}\cdots\mathbf{S}_k^{\alpha_k}$$

we need only check generators:

$$|\psi_C\rangle \in S \text{ iff } \mathbf{S}_\alpha|\psi_C\rangle = |\psi_C\rangle, \ \forall \mathbf{S}_\alpha \in \langle\mathbb{S}\rangle$$

Truth By Example

Example: $\mathrm{S}_{ex} = \langle \mathbf{XXXX}, \mathbf{ZZZZ}, \mathbf{XXII} \rangle$

$\mathbf{ZZZZ}|\psi_C\rangle = |\psi_C\rangle \Rightarrow$ computational basis states are even parity

$$|\psi_C\rangle \in \text{Span}\,[|0000\rangle, |0011\rangle, |0110\rangle, |0101\rangle, |1100\rangle, |1010\rangle, |1001\rangle, |1111\rangle]$$

$\mathbf{XXXX}|\psi_C\rangle = |\psi_C\rangle \Rightarrow$ use projector $\frac{1}{2}(\mathbf{IIII} + \mathbf{XXXX})$

$$|\psi_C\rangle \in \text{Span}\left[\frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle), \frac{1}{\sqrt{2}}(|0110\rangle + |1001\rangle), \frac{1}{\sqrt{2}}(|1100\rangle + |0011\rangle), \frac{1}{\sqrt{2}}(|1010\rangle + |0101\rangle)\right]$$

$\mathbf{XXII}|\psi_C\rangle = |\psi_C\rangle \Rightarrow$ use projector $\frac{1}{2}(\mathbf{IIII} + \mathbf{XXII})$

$$|\psi_C\rangle \in \text{Span}\left[\frac{1}{2}(|0000\rangle + |1111\rangle + |1100\rangle + |0011\rangle), \frac{1}{2}(|0110\rangle + |0110\rangle + |1010\rangle + |0101\rangle)\right]$$

Mysterious Extra Slide

Sizing Up The Code

Let S generated by $\mathbf{G}_1, \mathbf{G}_2, \ldots, \mathbf{G}_l$. Define the projectors onto the $+1$ eigenspace:

$$\mathbf{P}_i = \frac{1}{2}(\mathbf{I} + \mathbf{G}_i)$$

$$\mathbf{P}_i\mathbf{P}_j = \mathbf{P}_j\mathbf{P}_i, \quad \mathbf{P}_i^2 = \mathbf{P}_i, \quad \mathrm{Tr}[\mathbf{P}_i] = 2^{n-1}$$

Projector onto the codespace is

$$\mathbf{P} = \mathbf{P}_1\mathbf{P}_2 \cdots \mathbf{P}_l$$

$$\mathbf{P}^2 = \mathbf{P}, \quad \mathrm{dim}C = \mathrm{Tr}[\mathbf{P}]$$

$$\mathrm{Tr}\left[\mathbf{P}_1\mathbf{P}_2\right] = \frac{1}{4}\mathrm{Tr}\left[\mathbf{I} + \mathbf{G}_1 + \mathbf{G}_2 + \mathbf{G}_1\mathbf{G}_2\right] = 2^{n-2}$$

Not identity (generators!)

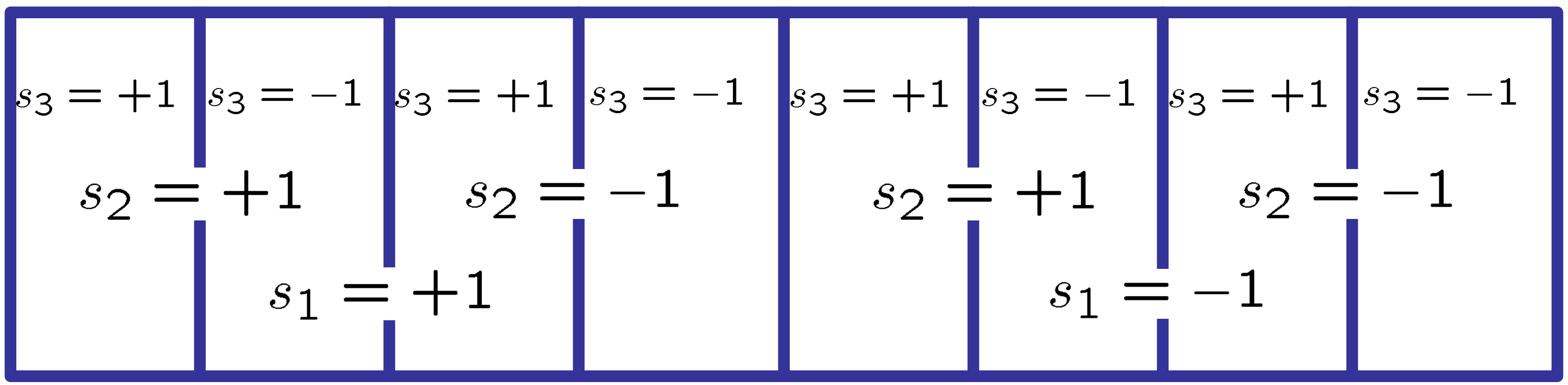
$$\mathbf{P} = \frac{1}{2^l} \sum_{\alpha_1, \alpha_2, \ldots, \alpha_l \in \{0,1\}} \mathbf{G}_1^{\alpha_1}\mathbf{G}_2^{\alpha_2} \cdots \mathbf{G}_l^{\alpha_l} \Rightarrow \mathrm{Tr}[\mathbf{P}] = 2^{n-l}$$

A stabilizer code generated by k elements has dimension 2^{n-k} (encodes $n-k$ qubits)

Stabilizer Subspaces

$\mathbf{G}_i|\psi\rangle = |\psi\rangle, \ \forall i$ defines a 2^{n-l} dimensional subspace.

Every unique $s_i = \pm 1$ defines, via $\mathbf{G}_i|\psi\rangle = s_i|\psi\rangle, \ \forall i$, a *uniques* subspace.



2^n dimensional Hilbert space

s_1	s_2	
$+1$	$+1$	← 2^{n-2} dimensional codespace
$+1$	-1	← 2^{n-2} dimensional codespace
-1	$+1$	← 2^{n-2} dimensional codespace
-1	-1	← 2^{n-2} dimensional codespace

$|s_1, s_2, \ldots, s_k, m\rangle$, m labels 2^{n-l} subspace

Stabilizer Nerror

Stabilizer $\mathbb{S}$ with generators $\mathbf{G}_1, \ldots, \mathbf{G}_k$ stabilizes a code of $n-l$ qubits. What errors does this code correct?

Suppose $\mathbf{E}$ anticommutes with *at least* one $\mathbf{G}_i$,

$$\{\mathbf{E}, \mathbf{G}_i\} = 0 \quad \mathbf{E}\mathbf{G}_i = -\mathbf{G}_i\mathbf{E}$$

then as before, for stabilized codewords,

$$\langle i|\mathbf{E}|j\rangle = \langle i|\mathbf{E}\mathbf{G}_i|j\rangle = -\langle i|\mathbf{G}_i\mathbf{E}|j\rangle = -\langle i|\mathbf{G}|j\rangle = 0$$

If $\mathbf{E}$ is in the stabilizer:

$$\langle i|\mathbf{E}|j\rangle = \langle i|j\rangle = \delta_{ij}$$

Given a set of errors $\mathbf{E}_a$, a stabilizer code S corrects these errors if the products $\mathbf{E}_a^\dagger \mathbf{E}_b$ either

(i) anticommute with at least one stabilizer generator

(ii) are in the stabilzer group

Surfing The Subspaces

If $\mathbf{E}_a$ anticommutes with $\mathbf{G}_i$,

$$\mathbf{G}_i(\mathbf{E}_a|s_1,\ldots,s_l,m\rangle) = -s_i(\mathbf{E}_a|s_1,\ldots,s_l,m\rangle)$$

$$|s_1,s_2,s_3,\ldots,s_l,m\rangle$$

$$\Big\downarrow\ \mathbf{E}_a$$

$$|\bar{s}_1,\bar{s}_2,\bar{s}_3,\ldots,\bar{s}_l,m\rangle$$

$$\bar{s}_i = s_i \text{ if } [\mathbf{G}_i,\mathbf{E}_a] = 0$$

$$\bar{s}_i = -s_i \text{ if } \{\mathbf{G}_i,\mathbf{E}_a\} = 0$$

$$|s_1=+1,s_2=+1,s_3=+1,s_4=+1,m\rangle$$

$\mathbf{E}_1$

$$|s_1=+1,s_2=+1,s_3=-1,s_4=-1,m\rangle$$

$\mathbf{E}_5$

$\mathbf{E}_2$

$$|s_1=+1,s_2=-1,s_3=+1,s_4=+1,m\rangle$$

$\mathbf{E}_7$

$$|s_1=-1,s_2=-1,s_3=-1,s_4=+1,m\rangle$$

Unsurfing The Subspaces

Measuring the generators $\mathbf{G}_i$ can be simultaneously performed (they commute!).

Measuring the generators $\mathbf{G}_i$ determines which $s_1, s_2, \ldots, s_k$ subspace the state is in.

Once the s_i are determined, returning to the codespace is simply rotating back to all $s_i = +1$?

$$|s_1 = +1, s_2 = +1, s_3 = +1, s_4 = +1, m\rangle$$

Error

$$|s_1 = +1, s_2 = -1, s_3 = -1, s_4 = +1, m\rangle$$

Fix ← Correct fix?

$$|s_1 = +1, s_2 = +1, s_3 = +1, s_4 = +1, m\rangle$$

Well yes, from iff condition

Examples

$\langle \mathtt{S} \rangle = \{\mathbf{ZZI}, \mathbf{IZZ}\}$ corrects single bit flip errors $\{\mathbf{E}_a\} = \{\mathbf{XII}, \mathbf{IXI}, \mathbf{IIX}\}$.

$\langle \mathtt{S} \rangle = \{\mathbf{XXI}, \mathbf{IXX}\}$ corrects single phase errors $\{\mathbf{E}_a\} = \{\mathbf{ZII}, \mathbf{IZI}, \mathbf{IIZ}\}$.

$$\langle \mathtt{S} \rangle = \{\mathbf{ZZI|III|III}, \mathbf{IZZ|III|III}, \mathbf{III|ZZI|III}, \mathbf{III|IZZ|III}, \mathbf{III|III|ZZI}, \mathbf{III|III|IZZ}, \mathbf{XXX|XXX|III}, \mathbf{III|XXX|XXX}\}$$

Shor code corrects single qubit errors $\{\mathbf{E}_a\} = \{\text{any single qubit error}\}$

$$\langle \mathtt{S} \rangle = \{\mathbf{XZZXI}, \mathbf{IXZZX}, \mathbf{XIXZZ}, \mathbf{ZXIXZ}\}$$

Five bit code corrects single qubit errors $\{\mathbf{E}_a\} = \{\text{any single qubit error}\}$

All encode one qubit.

$$\langle \mathtt{S} \rangle = \{\mathbf{ZZZZZZZZ}, \mathbf{XXXXXXXX}, \mathbf{IXIXYZYZ}, \mathbf{IXZYIXZY}, \mathbf{IYXZIYXZ}\}$$

8 qubit code corrects single qubit errors $\{\mathbf{E}_a\} = \{\text{any single qubit error}\}$

. . . but encodes 3 qubits!

Quantum Hamming Bound

Suppose all errors anticommute with at least one stabilizer generator (non-degenerate code).

n qubit code which corrects t or less errors and encodes k qubits.

If j error occur, $\binom{n}{j}$ ways to place these errors.

Each of the j errors can be one of 3 errors.

Total number of errors $\sum_{j=0}^{t} \binom{n}{j} 3^j$

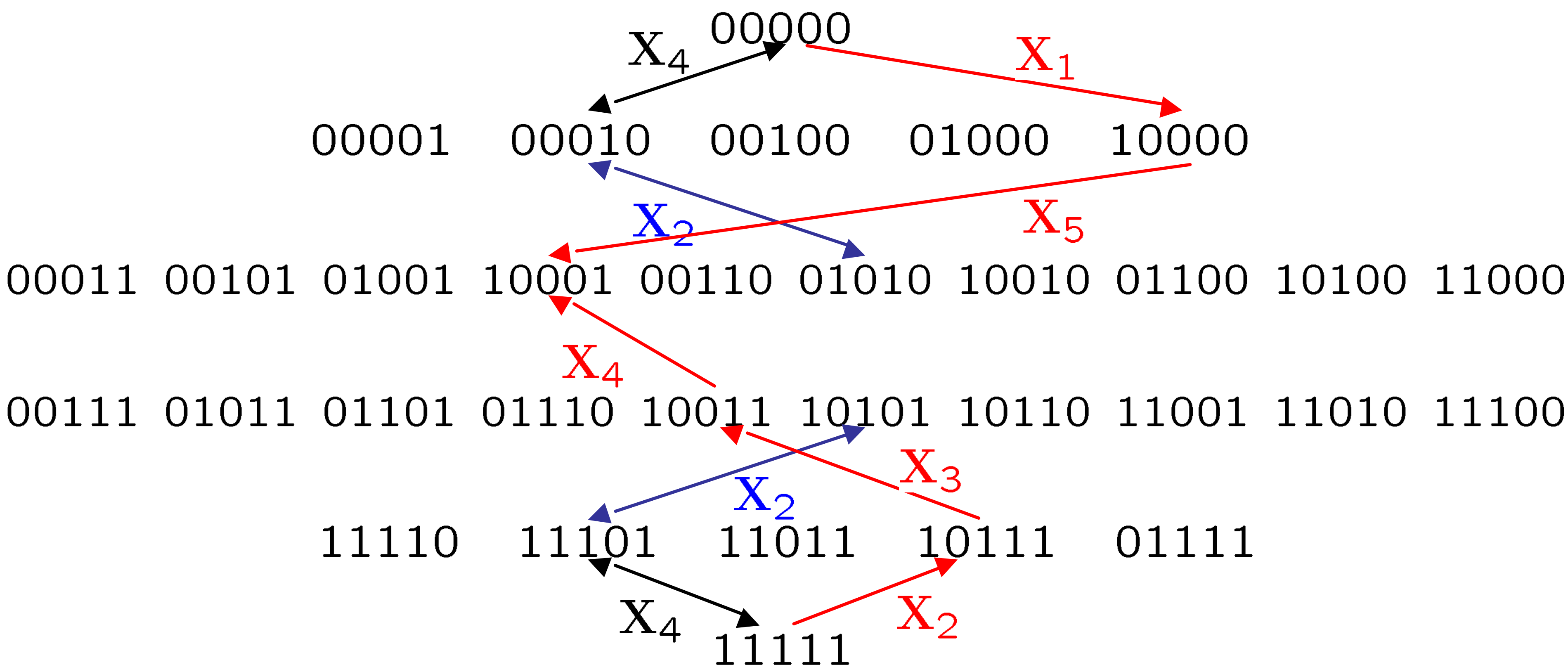
Each error gets its own subspace of dimension 2^k and all this must fit in the n qubit space

$$\sum_{j=0}^{t} \binom{n}{j} 3^j 2^k \leq 2^n$$

$$t = k = 1 \Rightarrow 2(1 + 3n) \leq 2^n \Rightarrow n \geq 5$$

Research Problem: BEAT THE QUANTUM HAMMING BOUND

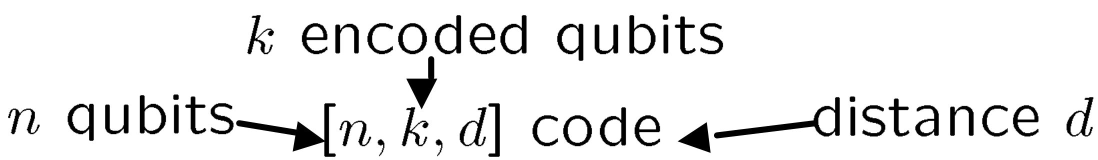
Detect Distinguishable Correct



Weight of Pauli group member = # non-identity tensor elements

Weight $(\mathbf{XYIZ})$=3

Smallest Weight for a code which takes codeword A to codeword $B = d$ =*distance*. A code which corrects t errormust be distance $2t + 1$.



Another Mysterious Extra Slide

Pauli Want a Qubit?

Return to our old friend the bit flip code: $\mathbf{S}_1 = \mathbf{ZZI}, \mathbf{S}_2 = \mathbf{IZZ}$

$$C = \text{Span}[|000\rangle, |111\rangle]$$

How do we perform encoded opeator on code?

Example: encoded "$\mathbf{X}$": $|000\rangle \Leftrightarrow |111\rangle$

$$\begin{aligned} \mathbf{XXX}|000\rangle &= |111\rangle & -\mathbf{XYY}|000\rangle &= |111\rangle & -\mathbf{YXY}|000\rangle &= |111\rangle \\ \mathbf{XXX}|111\rangle &= |000\rangle & -\mathbf{XYY}|111\rangle &= |000\rangle & -\mathbf{YXY}|111\rangle &= |000\rangle \end{aligned}$$

What is interesting about these operators $(\mathbf{XXX}, \mathbf{XYY}, \mathbf{YXY})$?

They commute with all stabilizer group elements . . .

. . . but are not themselves elements of the stabilizer group.

$$\mathcal{S} = \{\mathbf{III}, \mathbf{ZZI}, \mathbf{IZZ}, \mathbf{ZIZ}\}$$

$\underbrace{(\mathbf{XXX})\mathbf{S},}_{\text{Modulo the stabilizer } \mathbf{S}}$ $\mathbf{S} \in \mathcal{S}$ acts on code same as $\mathbf{XXX}$

Modulo the stabilizer **S**

$\mathbf{S}_1 = \mathbf{ZZI}, \mathbf{S}_2 = \mathbf{IZZ}$

Logical opeators: $\bar{\mathbf{X}} = \mathbf{XXX}$, $\bar{\mathbf{Y}} = \mathbf{YXX}$, $\bar{\mathbf{Z}} = \mathbf{ZII}$

$$\mathbf{XXX}|000\rangle = |111\rangle \qquad \mathbf{ZII}|000\rangle = |000\rangle$$

$$\mathbf{XXX}|111\rangle = |000\rangle \qquad \mathbf{ZII}|111\rangle = -|111\rangle$$

Modulo the stabilizer **S**

Feel free to randomly insert "Modulo the stabilizer **S**" into my sentence

Logical operators used to define "encoded" computational basis ± 1 eigenvalues of $\bar{Z}$.

Logical Operators for a Stabilizer Code

Given a stabilizer group $\mathbb{S}$, the set of Pauli operators which commute with all of the elements of $\mathbb{S}$, $\{\mathbf{L}|[\mathbf{L},\mathbf{S}], \forall \mathbf{S} \in \mathbb{S}\}$, for the logical operators (MTS) for the stabilizer code.

$\mathbf{S}(\mathbf{L}|\psi_C\rangle) = \mathbf{L}(\mathbf{S}|\psi_C\rangle) = (\mathbf{L}|\psi_C\rangle$: $\mathbf{L}$ preserves codespace.

Pauli Wants Lots of Qubits

Consider the code with a single stabilizer element on 3 qubits:

$$\mathbf{S}_1 = \mathbf{XXX}$$

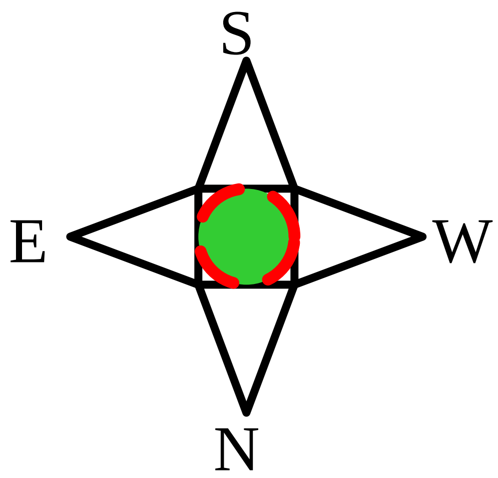
Should encode 2 qubits: what are the logical operators?

$$\begin{aligned} \bar{\mathbf{X}}_1 &= \mathbf{XII} \quad \bar{\mathbf{Z}}_1 = \mathbf{ZZI} \\ \bar{\mathbf{X}}_2 &= \mathbf{IIX} \quad \bar{\mathbf{Z}}_2 = \mathbf{IZZ} \end{aligned}$$

$$\bar{\mathbf{X}}_1\bar{\mathbf{Z}}_1 = -i\bar{\mathbf{Y}}_1 \qquad \bar{\mathbf{X}}_2\bar{\mathbf{Z}}_2 = -i\bar{\mathbf{Y}}_2 \qquad [\bar{\mathbf{P}}_1, \bar{\mathbf{P}}_2] = 0$$

But what is qubit 1 and qubit 2? Basis change renders notion of subsystem irrelevant: $\mathbf{U}^\dagger \mathbf{P}_i \mathbf{U}$

$$\begin{aligned} \bar{\mathbf{X}}_1 &= \mathbf{XIX} \quad \bar{\mathbf{Z}}_1 = \mathbf{ZZI} \\ \bar{\mathbf{X}}_2 &= \mathbf{IIX} \quad \bar{\mathbf{Z}}_2 = \mathbf{ZIZ} \end{aligned}$$



Who says north is up?

Real Life (I'm Plato) Example

The Calderbank-Shor-Steane $[7, 1, 3]$ code C_{CSS}

$$\begin{aligned} \langle \mathrm{S} \rangle \;\; = \;\; & \{\mathbf{XXXXIII}, \mathbf{XXIIXXI}, \mathbf{XIXIXIX}, \\ & \mathbf{ZZZZIII}, \mathbf{ZZIIZZI}, \mathbf{ZIZIZIZ}\} \end{aligned}$$

7 physical qubits, 1 encoded qubit, corrects all single qubit errors.

Logical Operators are

$$\bar{\mathbf{X}} = \mathbf{XXXXXXX} \quad \bar{\mathbf{Z}} = \mathbf{ZZZZZZZ}$$

(Nobody ever rights $\bar{\mathbf{Y}}$ because it is just $\bar{\mathbf{Y}} = i\bar{\mathbf{X}}\bar{\mathbf{Z}}$)

Clifford the Dog

Maintain codespace if

$$\mathbf{O}|\psi_C\rangle = \mathbf{O}\mathbf{S}_\alpha|\psi_C\rangle = \mathbf{S}_\beta\mathbf{O}|\psi_C\rangle, \ \forall\alpha,\beta$$

Operators $\mathbf{O}$ which permute the stabilizer elements:

$$\mathbf{O}\mathbf{S}_\alpha = \mathbf{S}_\beta\mathbf{O} \Rightarrow \mathbf{O}^\dagger\mathbf{S}_\beta\mathbf{O} = \mathbf{S}_\alpha$$

Logical operators are example of such $\mathbf{O}$ for which $\alpha = \beta$.

Example: $\mathbb{S} = \langle \mathbf{XXXX}, \mathbf{ZZZZ}\rangle$

$$\begin{aligned}(\mathbf{HHHH})\mathbf{XXXX}(\mathbf{HHHH}) &= \mathbf{ZZZZ} \\ (\mathbf{HHHH})\mathbf{YYYY}(\mathbf{HHHH}) &= \mathbf{YYYY}\end{aligned}$$

Example: C_{CSS} $\bar{\mathbf{X}} = \mathbf{XXXXXXX}$ $\bar{\mathbf{Z}} = \mathbf{ZZZZZZZ}$

$$\begin{aligned}(\mathbf{HHHHHHH})\bar{\mathbf{X}}(\mathbf{HHHHHHH}) &= \bar{\mathbf{Z}} \\ (\mathbf{HHHHHHH})\bar{\mathbf{Y}}(\mathbf{HHHHHHH}) &= -\bar{\mathbf{Y}}\end{aligned}$$

Stabilizer In Review

•Stabilizer group $\mathbb{S}$ is an abelian subgroup of the Pauli group with l generators $\mathbf{S}_\alpha$.

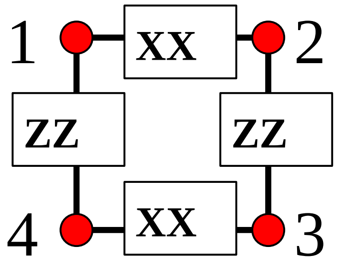
•The code corresponding to the stabilizer group $\mathbb{S}$ is the subspace spanned by the states which are stabilized by all of the stabilizer group operators $\mathbf{S}|\psi_C\rangle = |\psi_C\rangle$.

•The code corrects the set of errors $\{\mathbf{E}_a\}$ if $\mathbf{E}_a^\dagger \mathbf{E}_b$ anticommutes which at least one stabilizer element or $\mathbf{E}_a^\dagger \mathbf{E}_b$ is in the stabilizer.

•Logical operators for the encoded qubits are Pauli operators which commute with all the stabilizer elements but which are not themselves stabilizer elements.

Error Codes In Physical Systems

4 qubit system



$\mathbf{S}_1 = \mathbf{XXXX}$

$\mathbf{S}_2 = \mathbf{ZZZZ}$

$\mathbf{X}_1 = \mathbf{XXII},\ \mathbf{Z}_1 = \mathbf{IZZI}$

$\mathbf{X}_2 = \mathbf{IXXI},\ \mathbf{Z}_2 = \mathbf{ZZII}$

System Hamiltonian: $\mathbf{H} = \mathbf{XXII} + \mathbf{IIXX} + \mathbf{ZIIZ} + \mathbf{IZZI}$

Note symmetries: $[\mathbf{H}, \mathbf{S}_i] = [\mathbf{H}, \mathbf{X}_2] = [\mathbf{H}, \mathbf{Z}_2] = 0$

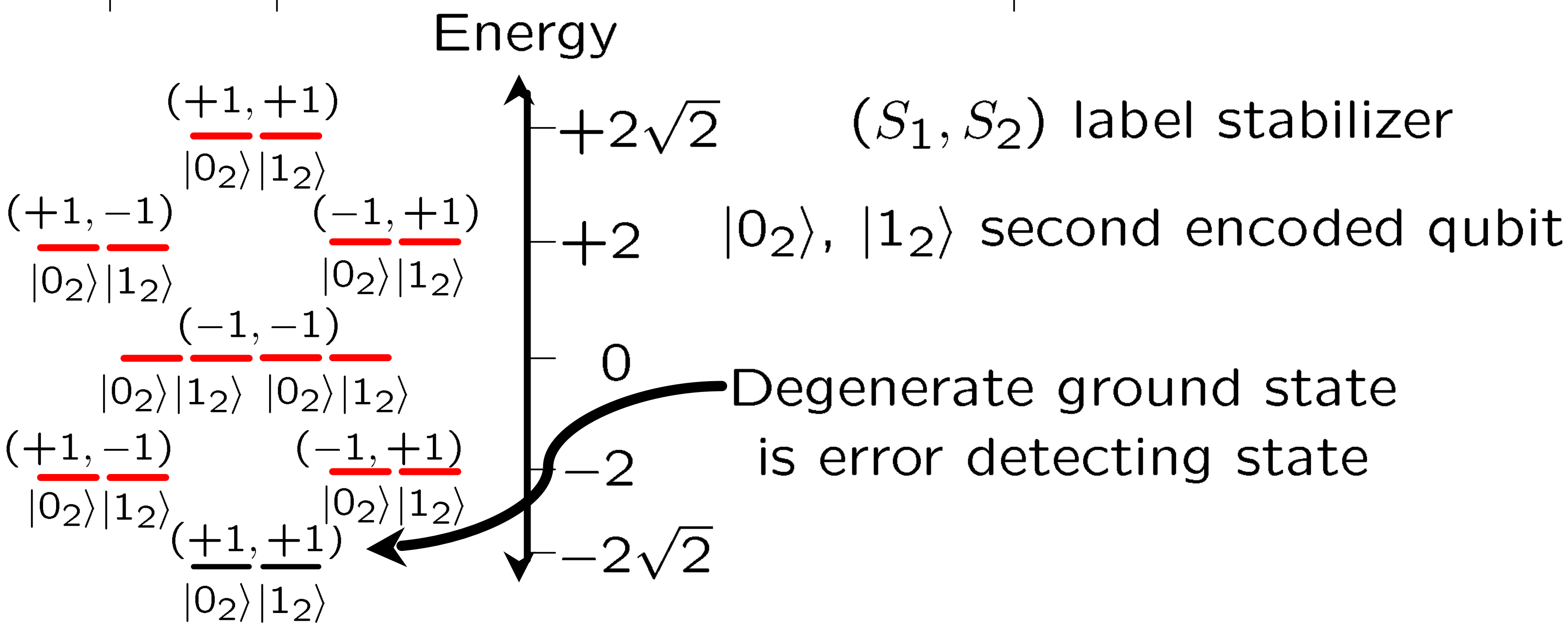
$$\mathbf{H} = \mathbf{X}_1(\mathbf{I} + \mathbf{S}_1) + \mathbf{Z}_1(\mathbf{I} + \mathbf{S}_2)$$

Does not involve 2nd encoded qubit.

$$\Downarrow$$

every energy level at least two-fold degnerate

S_1	S_2	$\mathbf{H} = \mathbf{X}_1(1+S_1) + \mathbf{Z}_1(1+S_2)$	Eigenvalues
$+1$	$+1$	$2(\mathbf{X}_1 + \mathbf{Z}_1)$	$\pm 2\sqrt{2}$
$+1$	-1	$2\mathbf{X}_1$	± 2
-1	$+1$	$2\mathbf{Z}_1$	± 2
-1	-1	$\mathbf{0}$	0

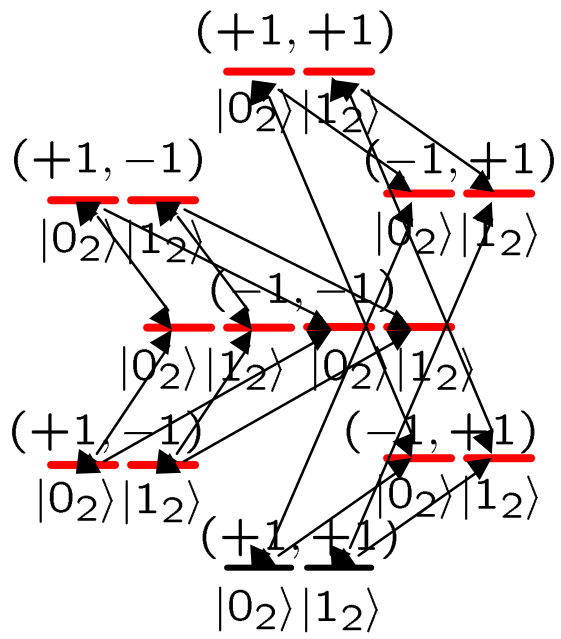


Supercoherence

$$\mathbf{H} = \mathbf{X}_1(1 + S_1) + \mathbf{Z}_1(1 + S_2)$$

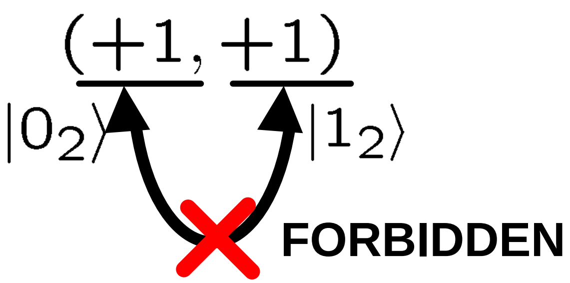
$\mathbf{S}_i\mathbf{P} = -\mathbf{P}\mathbf{S}_i$ for some i $\quad$ $\mathbf{P}$ single qubit Pauli operator

Effect of $\mathbf{ZIII}$

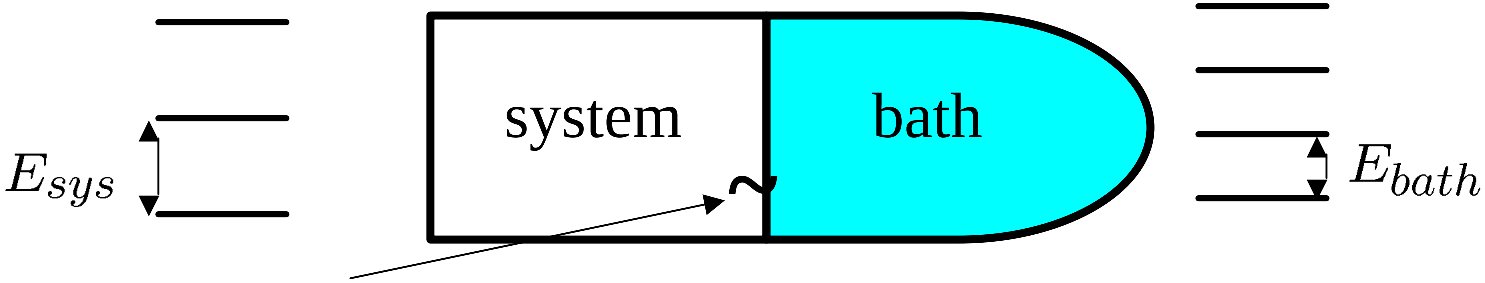


ALL single qubit errors take degenerate ground state to higher energy levels.

Single qubit errors change value of (S_1,S_2) and hence take ground state to higher energy level.



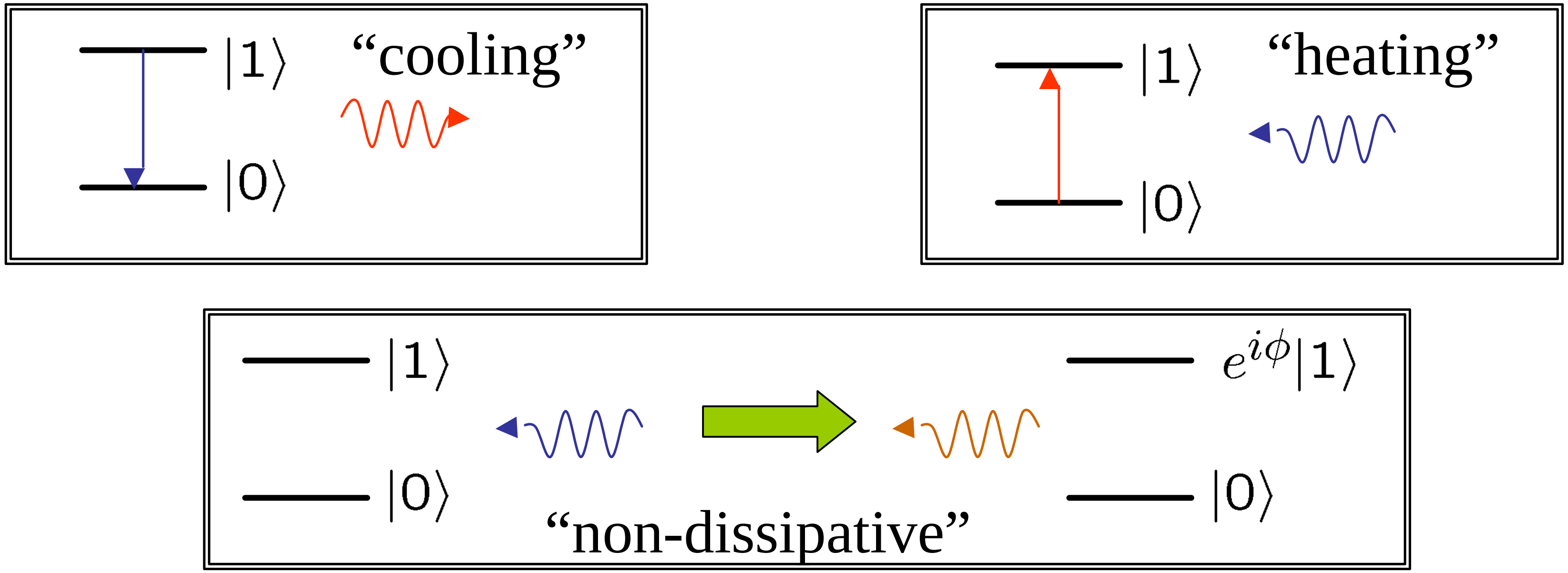
Making Decoherence Work Hard



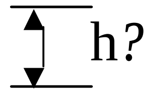
$\mathbf{H}_\mathrm{I}$ = perturbative system-bath coupling

$\mathbf{H}_\mathrm{I}$ energy small compared to E_{sys} and E_{bath} implies decoherence dominated by pathways that *conserve unperturbed energies*

(essentially rotating wave approximation)



two-level atom radiatively coupled to a thermal reservoir:



$$\frac{\partial \boldsymbol{\rho}}{\partial t} = -\bar{n}_{th}\frac{\gamma}{2}\left[\boldsymbol{\sigma}_-\boldsymbol{\sigma}_+\boldsymbol{\rho} + \boldsymbol{\rho}\boldsymbol{\sigma}_-\boldsymbol{\sigma}_+ - 2\boldsymbol{\sigma}_+\boldsymbol{\rho}\boldsymbol{\sigma}_-\right]$$

"heating"

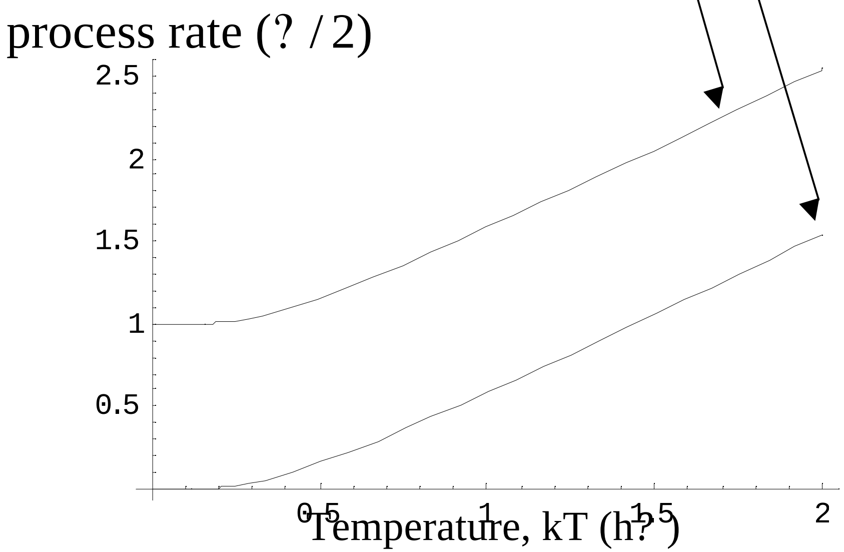
$$-(\bar{n}_{th}+1)\frac{\gamma}{2}\left[\boldsymbol{\sigma}_+\boldsymbol{\sigma}_-\boldsymbol{\rho} + \boldsymbol{\rho}\boldsymbol{\sigma}_+\boldsymbol{\sigma}_- - 2\boldsymbol{\sigma}_-\boldsymbol{\rho}\boldsymbol{\sigma}_+\right]$$

"cooling"

$$-\phi\left[\boldsymbol{\rho} - \boldsymbol{\sigma}_z\boldsymbol{\rho}\boldsymbol{\sigma}_z\right]$$

"non-dissipative"

$$\bar{n}_{th}\ ?\ \frac{1}{\exp ? \frac{h?}{kT} ? ? 1}$$



At low bath temperatures, heating disappears……..

Supercoherence

made all quantum errors "heating"!

Rant Mode On

Damnit!

PARENTAL ADVISORY EXPLICIT LYRICS

Only physicists understand timescales.

EVENTUALLY your hard drive will fail.

EVENTUALLY your PC will produce a hardware error.

Timescale for errors in classical computers are extremely long

Why? Why? Why? Why?

Physics. Physics? Physics. Physics!

Carver Mead Principle of Computation

There are distinct PHYSICAL reasons why classical computers are robust.

Corollary: Not all physical systems are created equal.

DO NOT FORGET OUR PRESILICON HISTORY

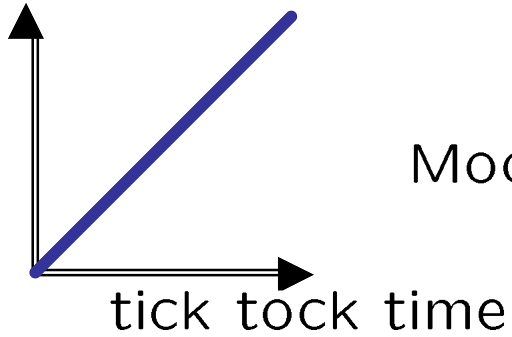
(NOT SO SUCCESSFUL)

Damnit! Part II

Rant Mode On

The computer revolution was (is) about scaling.

Moore's Law



Moore's Lucky Guess

Moore's Self-fulfilling Prophecy

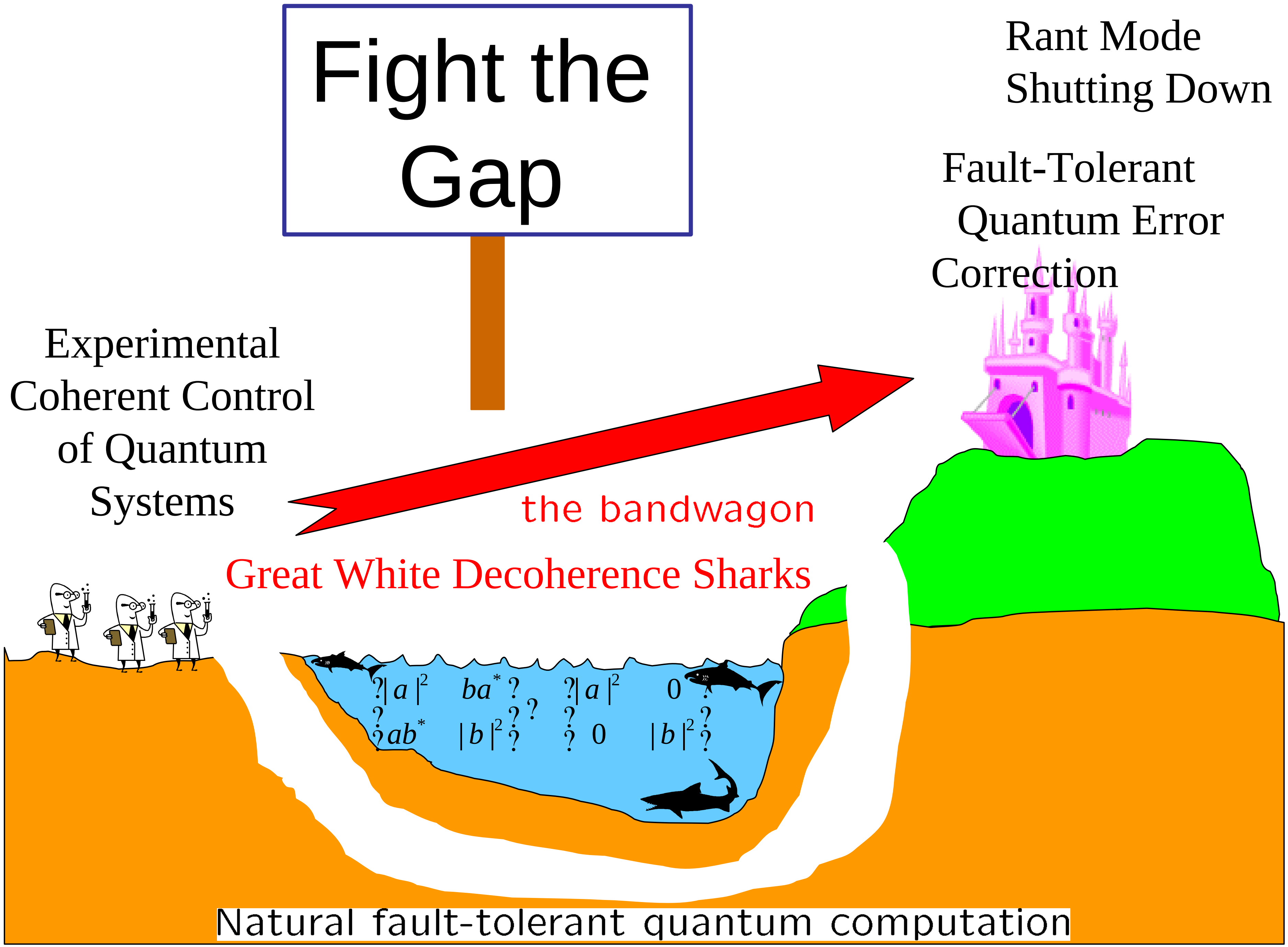
Classical computer scalability worked from top towards bottom

One could start from microscopic systems as the basic building blocks, but not technologically easy.

The dogma (meme): "quantum computer", being quantum (duh), must be microscopic.

Reasons to be a heretic: superconductivity, quantum hall effect, Alexi Kitaev's topological quantum codes, (this talk?),...

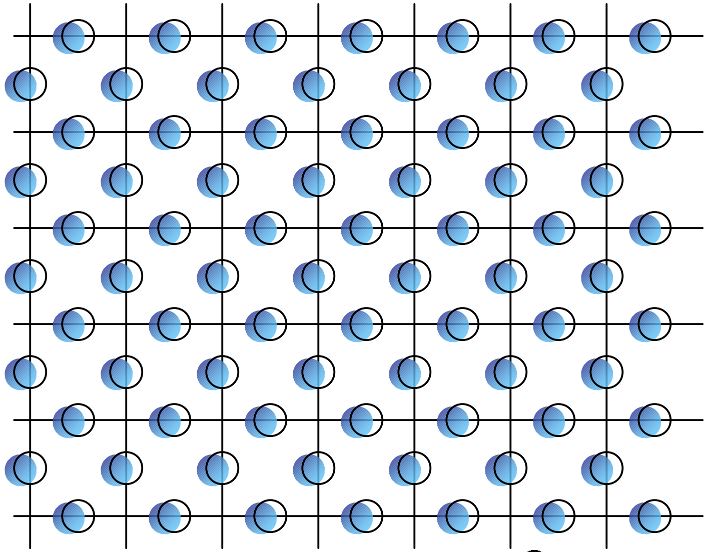
...Lesson of quantum error correction: not too different from classical error correction

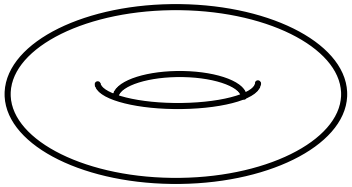

Fight the Gap
Rant Mode Shutting Down
Fault-Tolerant Quantum Error Correction
Experimental Coherent Control of Quantum Systems
the bandwagon
Great White Decoherence Sharks
Natural fault-tolerant quantum computation

Toric Codes

Kitaev 1997

n by n square lattice on a torus

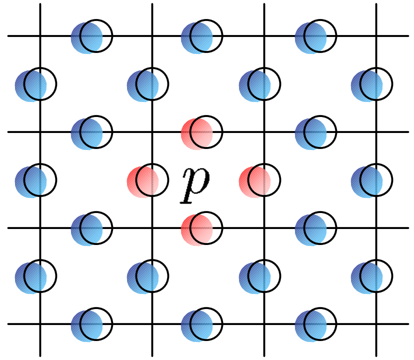




$2n^2$ qubits

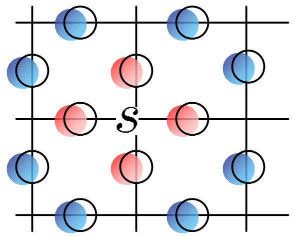
plaque operators

$\mathbf{B}_p = \mathbf{Z}_1\mathbf{Z}_2\mathbf{Z}_3\mathbf{Z}_4$



vertex operators

$\mathbf{A}_s = \mathbf{X}_1\mathbf{X}_2\mathbf{X}_3\mathbf{X}_4$



$[\mathbf{X} \otimes \mathbf{X}, \mathbf{Z} \otimes \mathbf{Z}] = 0$

plaque and vertex operators commute.

plaque and vertex operators have eigenvalues ± 1.

The toric Hamiltonian:

$$\mathbf{H} = -E_0 \left[\sum_e \mathbf{A}_e + \sum_p \mathbf{B}_p \right]$$

Ground state is stabilizer code state for $\mathbf{A}_e$, $\mathbf{B}_p$ operators!

There's a Code in my Hamiltonian

(…but no Wocket in my Pocket)

$$\mathbf{H} = -E_0 \left[\sum_e \mathbf{A}_e + \sum_p \mathbf{B}_p\right]$$

Since $\mathbf{A}_e$ and $\mathbf{B}_p$ commute we can simultaneously diagonalize these operators.

$$\mathbf{A}_e|\psi\rangle = A_e|\psi\rangle,\ \mathbf{B}_p|\psi\rangle = B_p|\psi\rangle \quad A_e, B_p \in \{+1, -1\}$$

$$|\psi\rangle = |A_1, \ldots, A_n, B_1, \ldots, B_n, \lambda\rangle$$

But $\prod_e \mathbf{A}_e = \mathbf{I}$, so $\prod_e A_e = +1$ $\qquad \prod_p \mathbf{B}_e = \mathbf{I}$, so $\prod_e B_e = +1$

$A_1, \ldots, A_n, B_1, \ldots, B_n$ span Hilbert space of dim $= 2^{2(n-1)}$

$$|\psi\rangle = |A_1, \ldots, A_n, B_1, \ldots, B_n, \lambda\rangle,\ \lambda \in \{00, 01, 10, 11\}$$

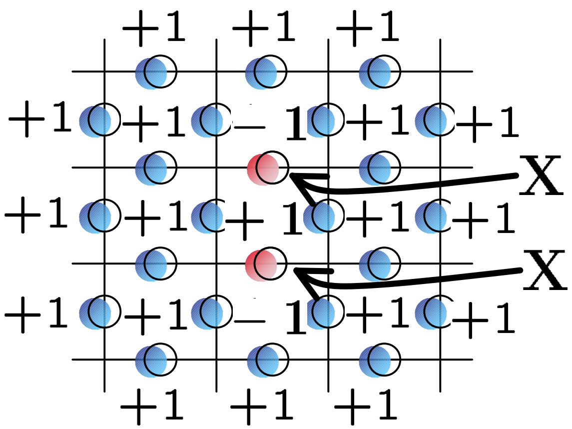
Ground state of $\mathbf{H} = -E_0 \left[\sum_e \mathbf{A}_e + \sum_p \mathbf{B}_p\right]$ has $A_i = B_i = +1$ and is four-fold degenerate (λ).

Anticommuting is Not a Conservative Agenda Item

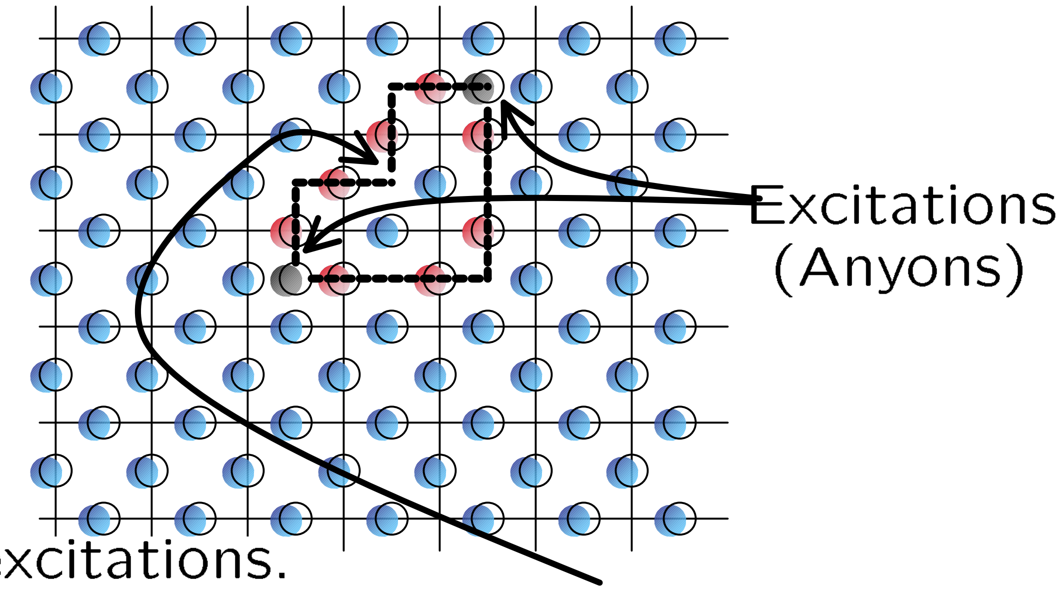
If $\{\mathbf{B}_p, \mathbf{E}\} = \mathbf{B}_p\mathbf{E} + \mathbf{E}\mathbf{B}_p = 0$ and $\mathbf{B}_p|\psi\rangle = B_p|\psi\rangle$, then

$$\mathbf{B}_p(\mathbf{E}|\psi\rangle) = -\mathbf{E}\mathbf{B}_p|\psi\rangle = -B_p(\mathbf{E}|\psi\rangle)$$

flips sign of ± 1 eigenvalue.



$$\mathbf{B}_p = \mathbf{Z}_1\mathbf{Z}_2\mathbf{Z}_3\mathbf{Z}_4$$

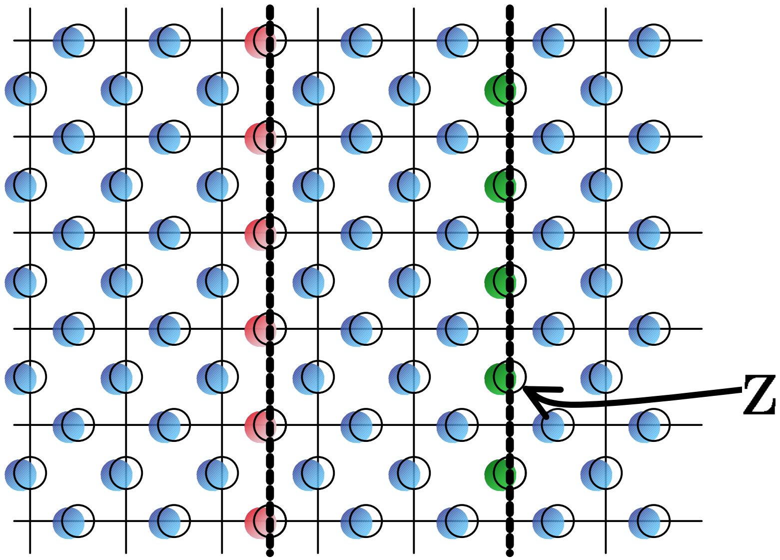


Measure all $\mathbf{B}_p$ to identify excitations.

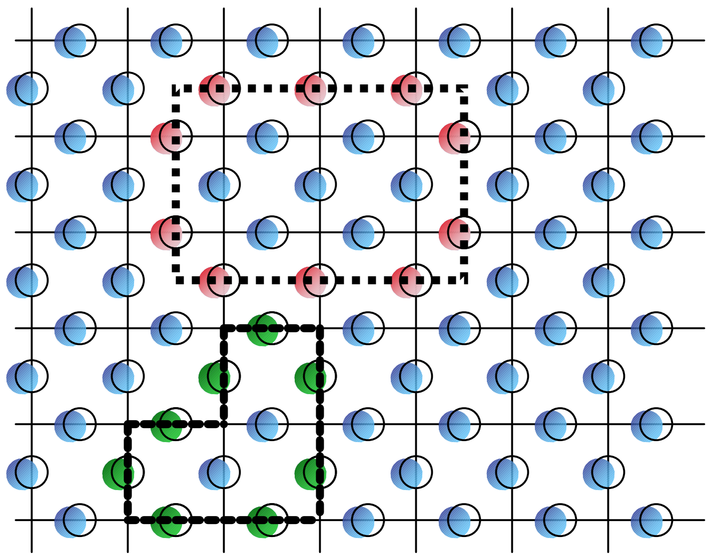
Fix by connecting excitations.

If path is contractable loop then error and fix = product of $\mathbf{A}_e$ s.t. $\prod|\psi\rangle = |\psi\rangle$

Mmmm...Topology



homologically nontrivial



homologically trivial

If path is non contractable loop then error and fix $\neq$ product of $\mathbf{A}_e$. Error!

Topological protection of the quantum information.

Toric Code

not Kitaev

Specific stabilizer code

Local Hamiltonian

Energy gap for excitations

More general models can be used for fault-tolerant quantum computation

Physical systems (Hamiltonians) which perform quantum error correction?

Review

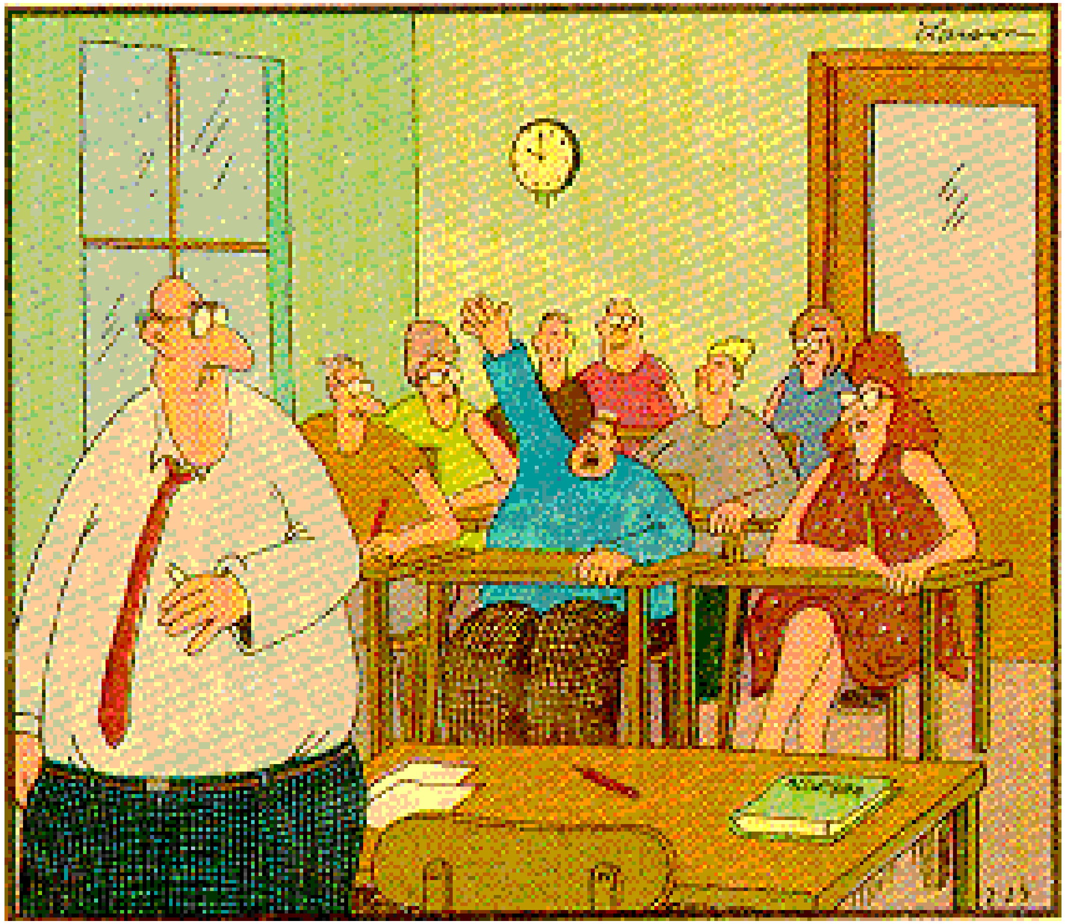
Pauli Stabilizer Codes

- number of encoded qubits
- errors that are corrected
- encoded operations
- quantum Hamming bound

Toric Code

- error correcting codes are energy eigenstates
- geometric protection of information
- natural fault-tolerance?

Fin



“sQuInT, may I be excused? My brain is empty.”

Pause While Plane Flies Over