# Short Course in Quantum Information Lecture 2


## Formal Structure of Quantum Mechanics

# Course Info

- All materials downloadable @ website

http://info.phys.unm.edu/~deutschgroup/DeutschClasses.html

- Syllabus

Lecture 1: Intro
Lecture 2: Formal Structure of Quantum Mechanics
Lecture 3: Entanglement
Lecture 4: Qubits and Quantum Circuits
Lecture 5: Algorithms
Lecture 6: Error Correction
Lecture 7: Physical Implementations
Lecture 8: Quantum Cryptography

# Lecture 1: Review

• Information is ***physical***: The ability of a machine to perform, e.g., computation is constrained by the laws of physics.

• Information: what we know.

- •Bayesian probability assignments based on prior knowledge.

• Quantum theory has its own logical rules.

- • Assign complex amplitude to quantum process.
- • Add amplitudes for ***indistinguishable*** *processes.*
- • Absolute square of amplitude gives probability of finding outcome --> ***Interference*** *of outcomes.*
- • Measurement "collapses" state assignment.
- • No local realistic description of "hidden variables".

# Course Info

- All materials downloadable @ website

http://info.phys.unm.edu/~deutschgroup/DeutschClasses.html

- Syllabus

Lecture 1: Intro
Lecture 2: Formal Structure of Quantum Mechanics
Lecture 3: Entanglement
Lecture 4: Qubits and Quantum Circuits
Lecture 5: Algorithms
Lecture 6: Error Correction
Lecture 7: Physical Implementations
Lecture 8: Quantum Cryptography

# The Ingredients of Quantum Theory

(Pure) States:
Assignments based on (maximal) knowledge of quantum system.

*probability*

Observables:
Measurable physical quantities

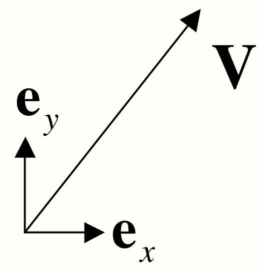
Measurement outcomes

***Intrinsically* stochastic**

Mathematical Structure (Hilbert space).
Vector space over complex numbers with inner product.

# Review:  Real vector space (Euclidean)

Linear superposition $\quad \mathbf{W} = c_1\mathbf{V} + c_2\mathbf{U}$




Basis vectors $\quad \mathbf{V} = V_x\mathbf{e}_x + V_y\mathbf{e}_y$

Inner product $\quad \|\mathbf{V}\|^2 \equiv \mathbf{V}\cdot\mathbf{V}$

$$\mathbf{Y}\cdot\mathbf{W} = \mathbf{Y}\cdot\left(c_1\mathbf{V} + c_2\mathbf{U}\right) = c_1\mathbf{Y}\cdot\mathbf{V} + c_2\mathbf{Y}\cdot\mathbf{U}$$

Orthonormal basis $\quad \mathbf{e}_x\cdot\mathbf{e}_x = \mathbf{e}_y\cdot\mathbf{e}_y = 1 \qquad \mathbf{e}_x\cdot\mathbf{e}_y = 0$

$$V_x = \mathbf{e}_x\cdot\mathbf{V} \qquad V_y = \mathbf{e}_y\cdot\mathbf{V} \qquad \|\mathbf{V}\|^2 = V_x^2 + V_y^2$$

# Complex Vector Space: Analytic Signals

Consider an oscillating electric vector, $\mathbf{E}(t) = E_0 \cos(\omega t)\mathbf{e}_x$

Intensity: $I = \frac{1}{T}\int_0^T |\mathbf{E}(t)|^2 = \frac{1}{2}E_0^2$

Complex signal: $\tilde{\mathbf{E}}(t) = E_0 \mathbf{e}_x \, e^{-i\omega t}$ $\quad \mathbf{E}(t) = \mathrm{Re}\left(\tilde{\mathbf{E}}(t)\right)$

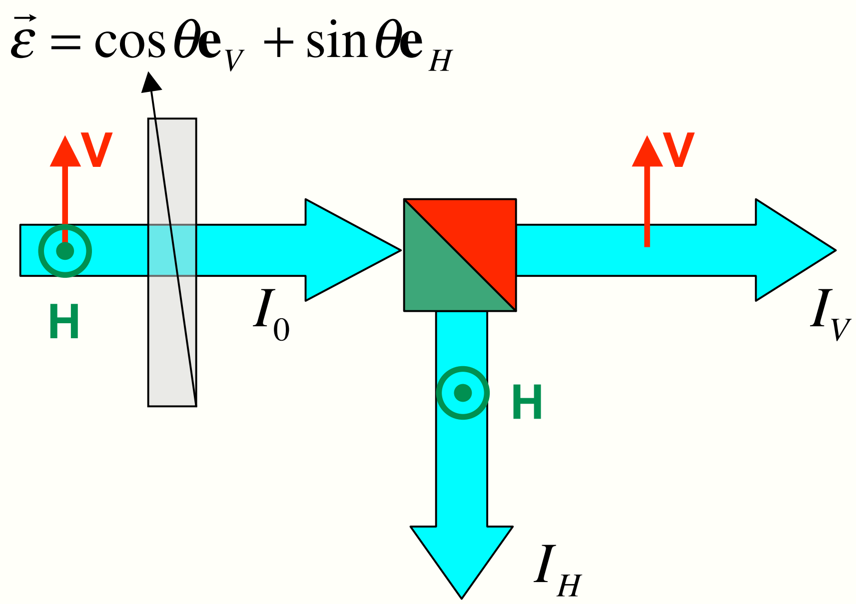
$$\tilde{\mathbf{E}}^* \cdot \tilde{\mathbf{E}} = \left(E_0 \mathbf{e}_x \, e^{+i\omega t}\right) \cdot \left(E_0 \mathbf{e}_x \, e^{-i\omega t}\right) = E_0^2 (e^{+i\omega t} \, e^{-i\omega t}) = E_0^2 \Longrightarrow I = \frac{1}{2}\tilde{\mathbf{E}}^* \cdot \tilde{\mathbf{E}}$$

Circular polarizaton: $\mathbf{E}(t) = E_0 \left(\cos(\omega t)\mathbf{e}_x + \sin(\omega t)\mathbf{e}_y\right) = \mathrm{Re}\left(\tilde{\mathbf{E}}(t)\right)$

$$\tilde{\mathbf{E}}(t) = E_0 \left(\mathbf{e}_x + i\mathbf{e}_y\right) e^{-i\omega t} \qquad I = \frac{1}{2}\tilde{\mathbf{E}}^* \cdot \tilde{\mathbf{E}} = \frac{1}{2} E_0^2 (\mathbf{e}_x - i\mathbf{e}_y) \cdot (\mathbf{e}_x + i\mathbf{e}_y) = E_0^2$$

Normalized polarization vector. $\quad \mathbf{e}_\pm \equiv \frac{\mathbf{e}_x \pm i\mathbf{e}_y}{\sqrt{2}} \qquad \left\|\mathbf{e}_\pm\right\|^2 = \mathbf{e}_\pm^* \cdot \mathbf{e}_\pm = 1 \qquad \mathbf{e}_+^* \cdot \mathbf{e}_- = 0$

# Motivation: Polarization Optics

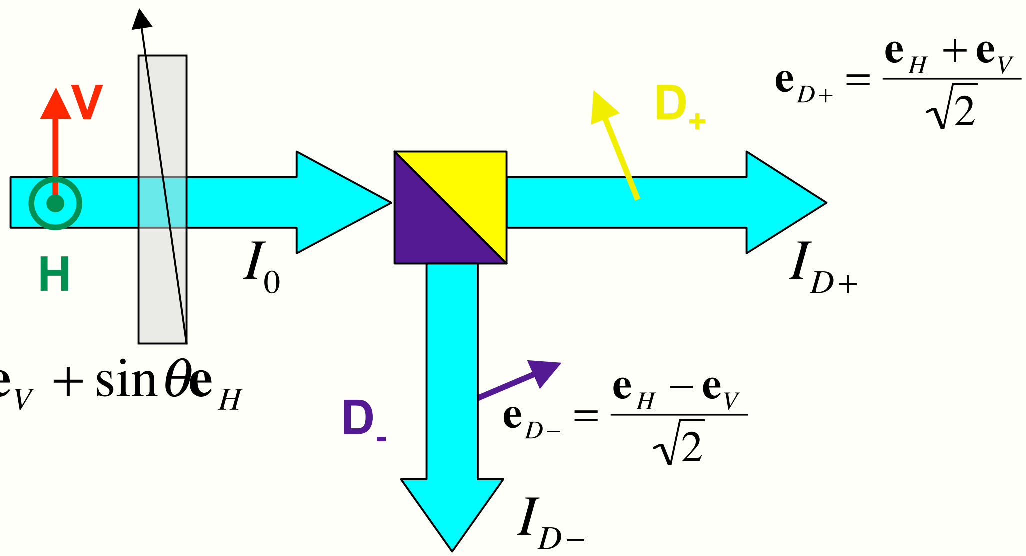



Malus's Law:

$$\mathbf{E}_V = \mathbf{e}_V\left(\mathbf{e}_V \cdot \mathbf{E}_0\right) = \mathbf{e}_V\left(\mathbf{e}_V \cdot \vec{\varepsilon}\right) E_0 = \mathbf{e}_V E_0 \cos\theta$$

$$I_V = \frac{1}{2}\mathbf{E}^* \cdot \mathbf{E} = \left|\mathbf{e}_V \cdot \vec{\varepsilon}\right|^2 \left(\frac{1}{2}\left|E_0\right|^2\right) = \cos^2\theta \, I_0$$

# Measurement in another basis (1)



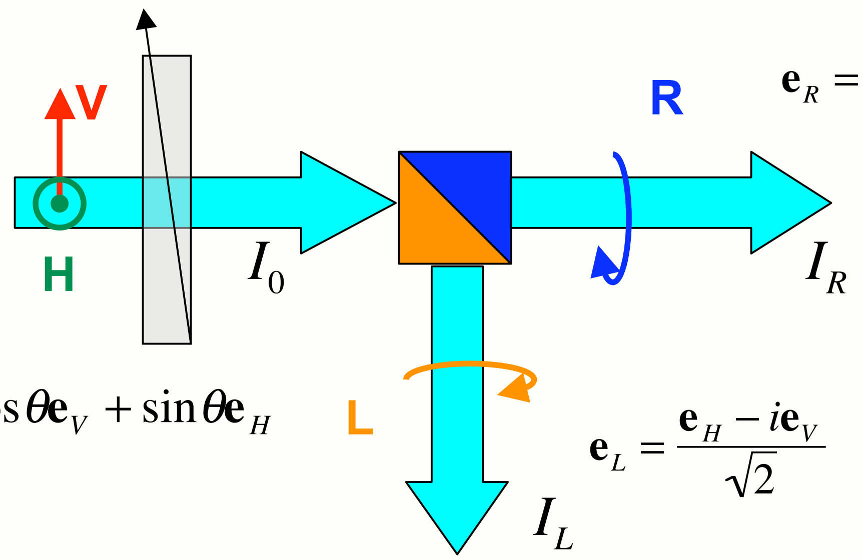

$$\vec{\varepsilon} = \cos\theta \mathbf{e}_V + \sin\theta \mathbf{e}_H$$

Malus's Law:

$$\mathbf{E}_{D\pm} = \mathbf{e}_{D\pm}\left(\mathbf{e}_{D\pm} \cdot \mathbf{E}_0\right) = \mathbf{e}_{D\pm}\left(\mathbf{e}_{D\pm} \cdot \vec{\varepsilon}\right) E_0 = \mathbf{e}_{D\pm}\left(\frac{\cos\theta \pm \sin\theta}{\sqrt{2}}\right) E_0$$

$$I_{D\pm} = \frac{1}{2}\mathbf{E}^*_{D\pm} \cdot \mathbf{E}_{D\pm} = \left|\mathbf{e}_{D\pm} \cdot \vec{\varepsilon}\right|^2 \left(\frac{1}{2}\left|E_0\right|^2\right) = \left(\frac{\cos\theta \pm \sin\theta}{\sqrt{2}}\right)^2 I_0$$

# Measurement in another basis (2)




$$\mathbf{e}_R = \frac{\mathbf{e}_H + i\mathbf{e}_V}{\sqrt{2}}$$

$$\vec{\varepsilon} = \cos\theta \mathbf{e}_V + \sin\theta \mathbf{e}_H$$

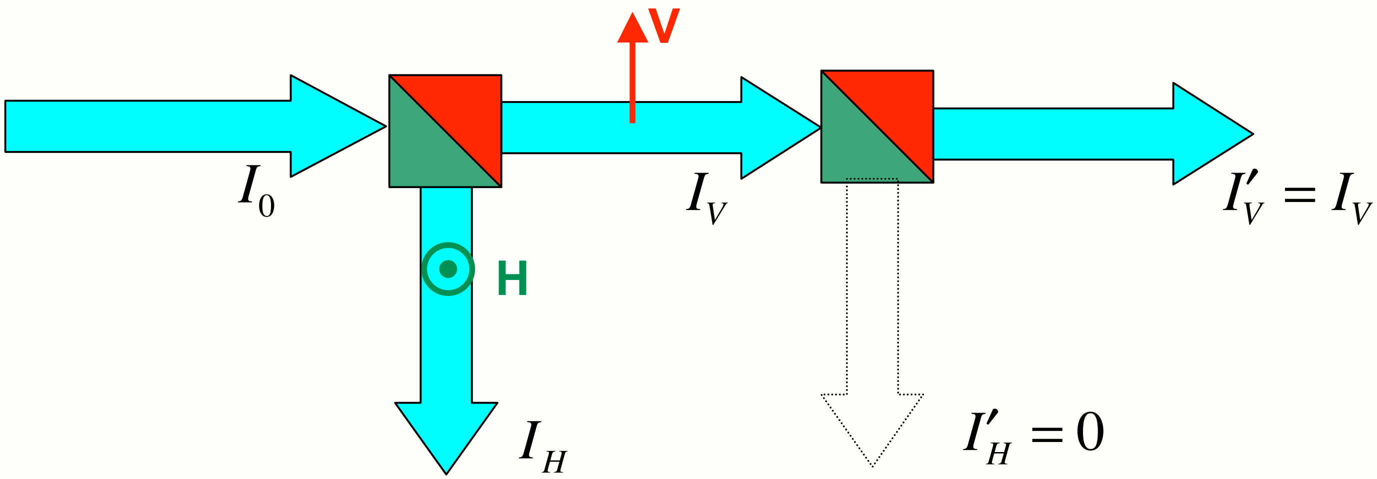
$$\mathbf{e}_L = \frac{\mathbf{e}_H - i\mathbf{e}_V}{\sqrt{2}}$$

## Malus's Law:

$$\mathbf{E}_{R,L} = \mathbf{e}_{R,L}\left(\mathbf{e}^*_{R,L} \cdot \mathbf{E}_0\right) = \mathbf{e}_{R,L}\left(\mathbf{e}^*_{R,L} \cdot \vec{\varepsilon}\right)E_0 = \mathbf{e}_{D\pm}\left(\frac{\cos\theta \mp i\sin\theta}{\sqrt{2}}\right)E_0 = \mathbf{e}_{D\pm}\left(\frac{e^{\mp i\theta}}{\sqrt{2}}\right)E_0$$

$$I_{R,L} = \frac{1}{2}\mathbf{E}^*_{R,L} \cdot \mathbf{E}_{R,L} = \left|\mathbf{e}_{R,L} \cdot \vec{\varepsilon}\right|^2 \left(\frac{1}{2}\left|E_0\right|^2\right) = \frac{1}{2}I_0$$

# Repeated measurement

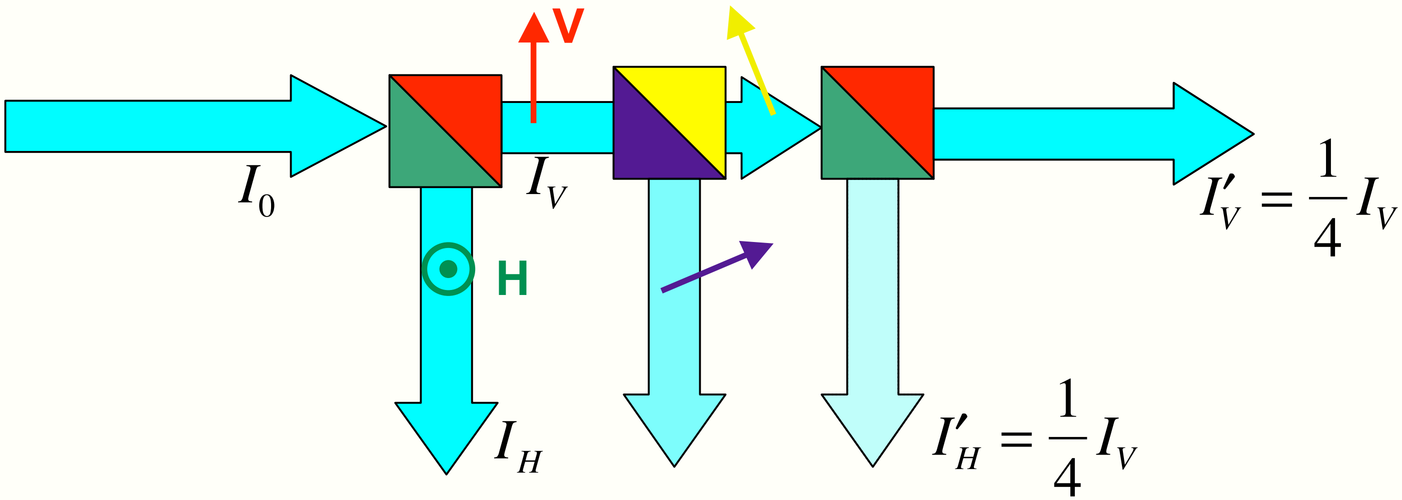



Malus's Law:

$$\mathbf{E}'_V = \mathbf{e}_V\left(\mathbf{e}_V \cdot \mathbf{E}_V\right) = \mathbf{e}_V E_V \qquad \mathbf{E}'_H = \mathbf{e}_H\left(\mathbf{e}_H \cdot \mathbf{E}_V\right) = 0$$

$$I'_V = \frac{1}{2}\mathbf{E}'^*_V \cdot \mathbf{E}'_V = I_V \qquad I'_H = 0$$

# Repeated measurement

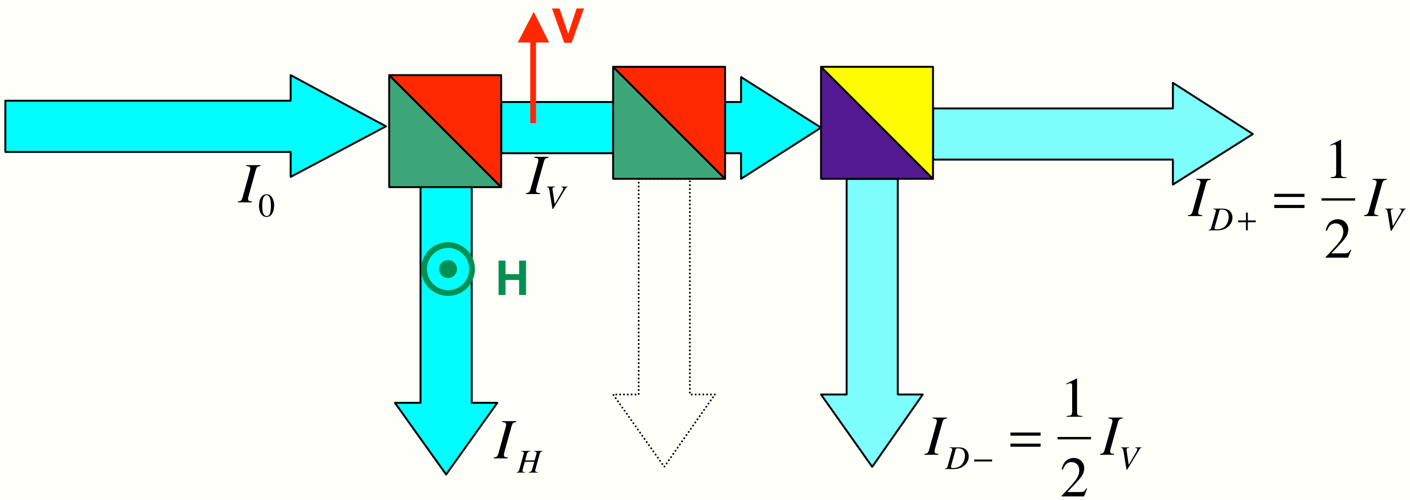



Malus's Law:

$$\mathbf{E}'_V = \mathbf{e}_V \left( \mathbf{e}_V \cdot \mathbf{E}_{D+} \right) = \mathbf{e}_V (\mathbf{e}_V \cdot \mathbf{e}_{D+})(\mathbf{e}_{D+} \cdot \mathbf{e}_V E_V) = \frac{1}{2} \mathbf{E}_V \Rightarrow I'_V = \frac{1}{4} I_V$$

$$\mathbf{E}'_H = \mathbf{e}_H \left( \mathbf{e}_H \cdot \mathbf{E}_{D+} \right) = \mathbf{e}_H (\mathbf{e}_H \cdot \mathbf{e}_{D+})(\mathbf{e}_{D+} \cdot \mathbf{e}_V E_V) = \frac{1}{2} \mathbf{E}_V \Rightarrow I'_H = \frac{1}{4} I_V$$

# Repeated measurements don't "commute"




Malus's Law:

$$\mathbf{E}_{D+} = \mathbf{e}_{D+}\left(\mathbf{e}_{D+} \cdot \mathbf{E}'_V\right) = \mathbf{e}_{D+}(\mathbf{e}_{D+} \cdot \mathbf{e}_V)(\mathbf{e}_V \cdot \mathbf{e}_V E_V) = \mathbf{e}_{D+} \frac{E_V}{\sqrt{2}} \Rightarrow I_{D+} = \frac{1}{2} I_V$$

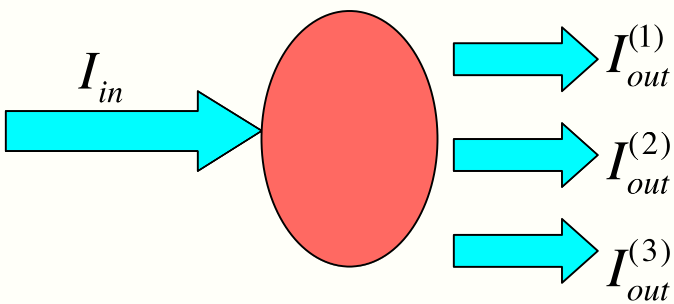
$$\mathbf{E}_{D-} = \mathbf{e}_{D-}\left(\mathbf{e}_{D-} \cdot \mathbf{E}'_V\right) = \mathbf{e}_{D-}(\mathbf{e}_{D-} \cdot \mathbf{e}_V)(\mathbf{e}_V \cdot \mathbf{e}_V E_V) = \mathbf{e}_{D-} \frac{E_V}{\sqrt{2}} \Rightarrow I_{D-} = \frac{1}{2} I_V$$

# From wave intensity to photon events

<u>Light beam = "stream of photon"</u> $I = \Phi \, \hbar\omega$

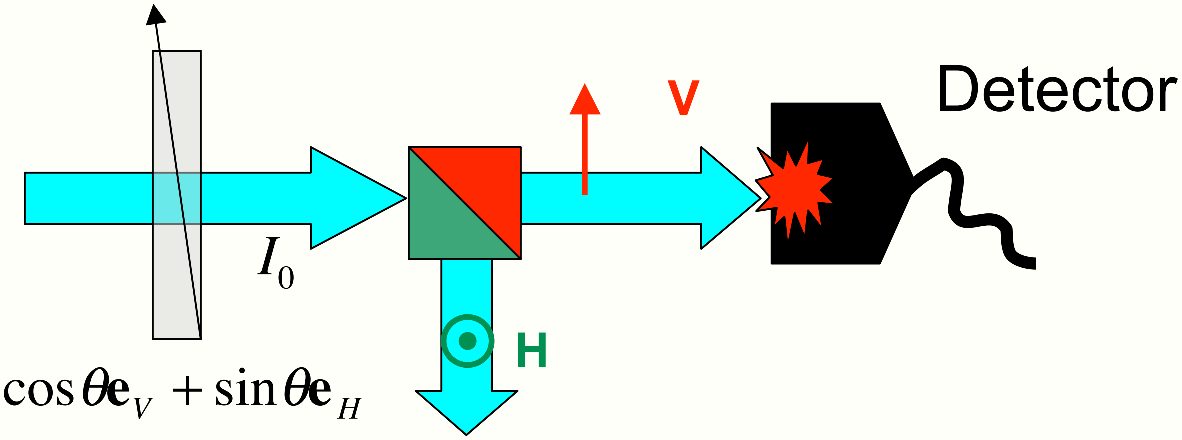
<u>In the "law of large numbers"</u>

| | |
|---|---|
| Fraction of intensity in given alternative | Probability of measuring photon in that alternative |




$$p_{out}^{(i)} = \frac{I_{out}^{(i)}}{I_{in}}$$

# From Malus's Law to Born's Rule




$$\vec{\varepsilon} = \cos\theta \mathbf{e}_V + \sin\theta \mathbf{e}_H$$

Malus's Law: $$I_V = \frac{1}{2}\mathbf{E}_V^* \cdot \mathbf{E}_V = \left|\mathbf{e}_V \cdot \vec{\varepsilon}\right|^2 \left(\frac{1}{2}\left|E_0\right|^2\right) = \cos^2\theta\, I_0$$

$$p_V = \frac{I_V}{I_0} = \left|\mathbf{e}_V \cdot \vec{\varepsilon}\right|^2 = \left|\cos\theta\right|^2$$

Generalize: Given photon prepared in polarization $\vec{\varepsilon}_{in}$,
probability of finding $\vec{\varepsilon}_{out}$, $\boxed{p_{out} = \left|\vec{\varepsilon}_{out}^{\,*} \cdot \vec{\varepsilon}_{in}\right|^2}$ **Born's Rule**

# From Photon Polarization to "Hilbert Space"

**Complex vector space of dimension *d*.**
**(Hilbert space can be infinite dimensional)**

## Dirac Notation:

"Kets" = vectors. $\left|V\right\rangle$

"Bras" = dual vectors. $\left\langle V\right|$

Inner product = braket $\left\langle V|V\right\rangle = \left\|V\right\|^2$

Like $\vec{\varepsilon}$ and $\vec{\varepsilon}^{\,*}$ for photon polarization, $\left\|\vec{\varepsilon}\right\|^2 = \vec{\varepsilon}^{\,*}\cdot\vec{\varepsilon}$

# Matrix Representation

Orthonormal basis $\left\{ \left| e_i \right\rangle \middle| i = 1, \ldots, d \right\}$ $\left\langle e_i \middle| e_j \right\rangle = \delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

$$\left| V \right\rangle = \sum_i V_i \left| \mathbf{e}_i \right\rangle \qquad V_i = \left\langle e_i \middle| V \right\rangle$$

ket =
column vector

$$\left| V \right\rangle \doteq \begin{bmatrix} V_1 \\ \vdots \\ V_d \end{bmatrix}$$

bra =
row vector conjugated (adjoint)

$$\left\langle V \right| \doteq \begin{bmatrix} V_1^* & \cdots & V_d^* \end{bmatrix} = \left| V \right\rangle^\dagger$$

Inner product via
matrix multiplication

$$\left\langle U \middle| V \right\rangle = \begin{bmatrix} U_1^* & \cdots & U_d^* \end{bmatrix} \begin{bmatrix} V_1 \\ \vdots \\ V_d \end{bmatrix} = \sum_i U_i^* V_i$$

# Linear Operators

<u>Maps</u> $\hat{A}|U\rangle = |V\rangle \qquad \hat{A}|W\rangle = |Y\rangle$

<u>Linear</u> $\hat{A}\left(a|U\rangle + b|W\rangle\right) = a\hat{A}|U\rangle + b\hat{A}|V\rangle = a|V\rangle + b|Y\rangle$

<u>Matrix Representation</u>

$$\hat{A} \doteq \begin{bmatrix} A_{11} & \cdots & A_{1d} \\ \vdots & \ddots & \vdots \\ A_{d1} & \cdots & A_{dd} \end{bmatrix}$$

$$A_{ij} = \langle e_i | \left( \hat{A} | e_j \rangle \right) \equiv \langle e_i | \hat{A} | e_j \rangle$$

$$\langle e_1 | \hat{A} | e_2 \rangle = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix} = A_{12}$$

# Special Operator Relations

Characteristic vectors/values (eigenvectors, eigenvalues)

$$\hat{A}|V_a\rangle = a|V_a\rangle$$

Hermitian operator: $\hat{A}^\dagger = \hat{A}$

Real eigenvalues,
Orthogonal eigenvectors

Unitary: $\hat{U}^\dagger \hat{U} = \hat{I}$

Preserves the inner product

$$\hat{U}|V\rangle = |V'\rangle \qquad \hat{U}|W\rangle = |W'\rangle$$

$$\langle V'|W'\rangle = \langle V|\hat{U}^\dagger \hat{U}|W\rangle = \langle V|W\rangle$$

# The Postulates of QM (I)

- A "pure state" of the system, representing some maximal knowledge, is a normalized ket in Hilbert space $|\psi\rangle$.

$$\langle\psi|\psi\rangle = \|\psi\|^2 = 1$$

- To every observable we associate a Hermitian operator $\hat{A}$.

- The possible values of an observable that one finds in a measurement are its *eigenvalues,* $\{a\}$.

- The probability of finding a value $a$ is given by the square of the amplitude in that eigenvector.

$$P_{a|\psi} = \left|\langle e_a|\psi\rangle\right|^2$$

# Notes

- The kets $|\psi\rangle$ and $|\psi'\rangle = e^{i\phi}|\psi\rangle$ are the *same state.*

$$P_{a|\psi'} = \left|\langle e_a|\psi'\rangle\right|^2 = \left|\langle e_a|\psi\rangle\right|^2 = P_{a|\psi}$$

- The *relative phase* in a superposition is *very* important.

$$|\psi\rangle = r_1 e^{i\phi_1}|u_1\rangle + r_2 e^{i\phi_2}|u_2\rangle$$

$$P_{a|\psi} = \left|\langle e_a|\psi\rangle\right|^2 = r_1^{\,2}\left|\langle e_a|u_1\rangle\right|^2 + r_2^{\,2}\left|\langle e_a|u_2\rangle\right|^2$$
$$+ r_1 r_2\left(e^{i(\phi_1-\phi_2)}\langle e_a|u_1\rangle^*\langle e_a|u_2\rangle + e^{-i(\phi_1-\phi_2)}\langle e_a|u_1\rangle\langle e_a|u_2\rangle^*\right)$$

$$= p_1 p_{a|1} + p_2 p_{a|2} + \text{inteference}$$

# Notes

• The set of possible measurement outcomes for an observable "span" the space $\Longrightarrow$ complete set.

$$|\psi\rangle = \sum_a c_a |e_a\rangle \qquad c_a = \langle e_a|\psi\rangle \qquad p_{a|\psi} = |\langle e_a|\psi\rangle|^2 = |c_a|^2$$

$$\langle\psi|\psi\rangle = \sum_a |c_a|^2 = \sum_a p_{a|\psi} = 1$$ Total probability = 1

$$\langle A\rangle_\psi = \sum_a a\, p_{a|\psi} = \sum_a a\, |\langle e_a|\psi\rangle|^2 = \langle\psi|\hat{A}|\psi\rangle$$ Expectation value

• Not all operators share the same eigenvectors

$\Longrightarrow$ *Impossible* to measure all observables simultaneously. Measurement of one observable disturbs the other.

**Heisenberg Uncertainty principle**

# Postulates of QM (II)

• After a measurement, the state of the system is updated to the eigenvector associated with the measurement outcome.

$$|\psi\rangle \underset{a}{\Rightarrow} |e_a\rangle \qquad \text{(Bayesian)}$$

• In the absence of any measurement, a *closed system* evolves according to a unitary map (preserving inner products).

$$|\psi(t+\tau)\rangle = \hat{U}(\tau)|\psi(t)\rangle$$

• Physics determines the dynamics $\frac{d}{dt}\hat{U}(t) = \frac{-i}{\hbar}\hat{H}\hat{U}(t)$

**The (time dependent) Schrödinger Equation**

# Example: Photon Polarization

- State: Normalized complex polarization vector $\vec{\varepsilon},\ \vec{\varepsilon}^* \cdot \vec{\varepsilon} = 1$

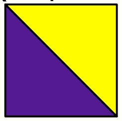
- Orthonormal bases: $\{\mathbf{e}_H, \mathbf{e}_V\}$ $\{\mathbf{e}_{D_+}, \mathbf{e}_{D-}\}$ $\{\mathbf{e}_R, \mathbf{e}_L\}$

Let us call {H,V} the “standard basis”

$$\mathbf{e}_H = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \ \mathbf{e}_V = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \mathbf{e}_{D+} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \ \mathbf{e}_{D-} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \mathbf{e}_R = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix} \ \mathbf{e}_L = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -i \end{bmatrix}$$

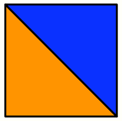
- Three (incompatible) observables:

($D_+$,$D_-$)



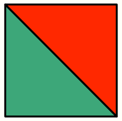
$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

(R,L)



$$Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

(H,V)



$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

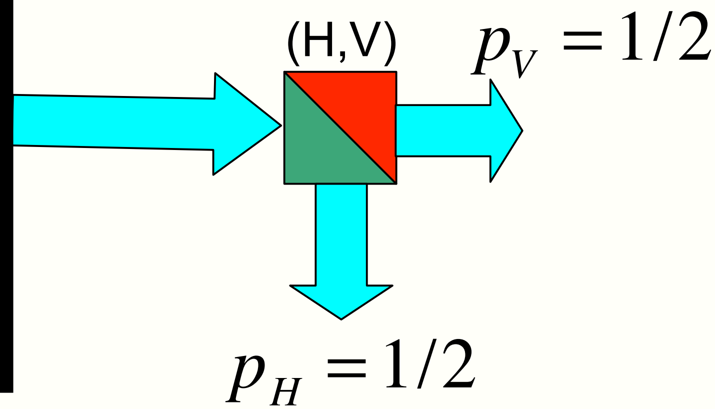
Pauli matrices with eigenvalues ±1

# Incomplete Information

Suppose someone prepares a photon by flipping a coin.

State?

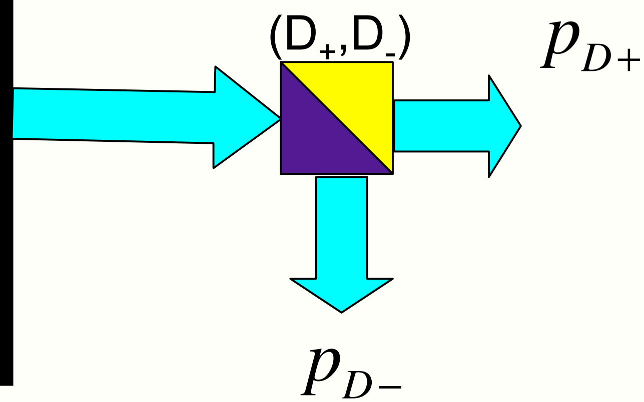
Statistical Mixture:
50% H, 50% V



# Incomplete Information

State?

Statistical Mixture:
50% H, 50% V




$$p_{D+} = p_H \, p_{D+|H} + p_V \, p_{D+|V}$$

$$p_{D+} = \frac{1}{2}\frac{1}{2} + \frac{1}{2}\frac{1}{2} = \frac{1}{2}$$

Unpolarized!

**Density Operator**

# Quantum Computing vs. Wave Computing

- Analog vs. Digital.
  - Quantum phenomena involve *discrete events* (e.g.  photon detection).
  - Wave and Particle ⇔ Analog and Digital.
  - Possibility of error correction.

- Physical space vs. Hilbert space
  - Modes of field require physical resources.
  - Exponential number of modes not scalable.
  - Quantum mechanics allows exponentially large dimension with polynomial physical resources.
  - ***Entangled states!***

# Summary

- Quantum Mechanics predicts the outcomes of experiments.
- States = Vectors in a complex linear space.
- Observables = Matrices (operators) on the space
  - Eigenvalues = possible measurement outcomes.
  - Eigenvectors = the state after the measurement is done.
- Probability of finding a measurement value = square of complex amplitude (Born rule).
- In a closed system, the system dynamics described by unitary matrices.
- Quantum “coherent superpositions” differ from “statistical mixtures”. Noise can “decoher” a system.
- Quantum computing not equivalent to wave computing.