## Short Course in Quantum Information Lecture 3

## Entanglement

## Course Info

- All materials downloadable @ website http://info.phys.unm.edu/~deutschgroup/DeutschClasses.html
- Syllabus

Lecture 1: Intro
Lecture 2: Formal Structure of Quantum Mechanics
Lecture 3: Entanglement
Lecture 4: Qubits and Quantum Circuits
Lecture 5: Algorithms
Lecture 6: Error Correction
Lecture 7: Physical Implementations
Lecture 8: Quantum Cryptography


## Postulates of QM

- A (pure) state of the system describing our knowledge of the system is given by a vector in Hilbert space, $|\psi\rangle$.
- A physical observable is a Hermitian linear operator $\hat{A}$ whose (real) eigenvalues $\{a\}$ determine the possible measurement outcomes.
- Given state $|\psi\rangle$, and measurement of $\hat{A}$, the probability of finding eigenvalue $a$ is given by,

$$
p_{a l \mid \psi}=\langle\langle a \mid \psi\rangle\rangle^{2}
$$

where $|a\rangle$ is the eigenvector of $\hat{A}$.

- Upon finding value $a$ the state "collapses", $|\psi\rangle \underset{a}{a}|a\rangle$
- In a closed system the state dynamics is determined by the Schrödinger equation ---> Unitary map that preserves in product.


## Example: Photon Polarization (I)

$$
\vec{\varepsilon}=\cos \theta \mathbf{e}_{H}+\sin \theta \mathbf{e}_{V}=\cos \theta|H\rangle+\sin \theta|V\rangle=\left[\begin{array}{c}
\cos \theta \\
\sin \theta
\end{array}\right]
$$


$\hat{Z}=\left[\begin{array}{cc}1 & 0 \\ 0 & -1\end{array}\right]$
Eigenvalues/vectors

$$
\begin{aligned}
& +1 \Rightarrow|H\rangle=\left[\begin{array}{l}
1 \\
0
\end{array}\right] \\
& -1 \Rightarrow|V\rangle=\left[\begin{array}{l}
0 \\
1
\end{array}\right]
\end{aligned}
$$

$$
\langle\hat{Z}\rangle_{\theta}=\left\langle+1_{\theta}\right| \hat{Z}\left|+1_{\theta}\right\rangle=\cos ^{2} \theta-\sin ^{2} \theta=\cos (2 \theta)
$$

## Example: Photon Polarization (II)

- State: Normalized complex polarization vector $\vec{\varepsilon}, \vec{\varepsilon}^{*} \cdot \vec{\varepsilon}=1$
- Orthonormal bases: $\left\{\mathbf{e}_{H}, \mathbf{e}_{V}\right\} \quad\left\{\mathbf{e}_{D_{+}+}, \mathbf{e}_{D-}\right\} \quad\left\{\mathbf{e}_{R}, \mathbf{e}_{L}\right\} \quad$ Let us call $\{\mathrm{H}, \mathrm{V}\}$ the "standard basis"

$$
\mathbf{e}_{H}=\left[\begin{array}{l}
1 \\
0
\end{array}\right] \mathbf{e}_{V}=\left[\begin{array}{l}
0 \\
1
\end{array}\right] \quad \mathbf{e}_{D+}=\frac{1}{\sqrt{2}}\left[\begin{array}{l}
1 \\
1
\end{array}\right] \mathbf{e}_{D-}=\frac{1}{\sqrt{2}}\left[\begin{array}{c}
1 \\
-1
\end{array}\right] \quad \mathbf{e}_{R}=\frac{1}{\sqrt{2}}\left[\begin{array}{l}
1 \\
i
\end{array}\right] \quad \mathbf{e}_{L}=\frac{1}{\sqrt{2}}\left[\begin{array}{c}
1 \\
-i
\end{array}\right]
$$

- Three (incompatible) observables:


$$
X=\left[\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right]
$$

$Y=\left[\begin{array}{cc}0 & -i \\ i & 0\end{array}\right]$
$Z=\left[\begin{array}{cc}1 & 0 \\ 0 & -1\end{array}\right]$

Pauli matrices with eigenvalues $\pm 1$

## Eigenstates of General Linear Polarization Analyzer

Define:


$$
\hat{\sigma}_{\phi} \equiv \cos 2 \phi \hat{Z}+\sin 2 \phi \hat{X}=\cos 2 \phi\left[\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right]+\sin 2 \phi\left[\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right]=\left[\begin{array}{cc}
\cos 2 \phi & \sin 2 \phi \\
\sin 2 \phi & -\cos 2 \phi
\end{array}\right]
$$

Eigenvectors/values

$$
\hat{\sigma}_{\phi}\left| \pm 1_{\phi}\right\rangle= \pm 1\left| \pm 1_{\phi}\right\rangle \quad\left|+1_{\phi}\right\rangle=\left[\begin{array}{c}
\cos \phi \\
\sin \phi
\end{array}\right] \quad\left|-1_{\phi}\right\rangle=\left[\begin{array}{c}
-\sin \phi \\
\cos \phi
\end{array}\right]
$$

Every linear polarization is +1 eigenvector of some $\hat{\sigma}_{\phi}$

$$
\left|e_{\phi}\right\rangle=\cos \phi|H\rangle+\sin \phi|V\rangle=\left|+1_{\phi}\right\rangle \quad 0 \leq \phi<\pi
$$

## Basic Measurement Statistics (I)

$$
\vec{\varepsilon}=\cos \theta \mathbf{e}_{H}+\sin \theta \mathbf{e}_{V}
$$

$$
\left\langle\hat{\sigma}_{\phi}\right\rangle_{\theta}=\left\langle+1_{\theta}\right| \hat{\sigma}_{\phi}\left|+1_{\theta}\right\rangle=\cos ^{2}(\phi-\theta)-\sin ^{2}(\phi-\theta)=\cos [2(\phi-\theta)]
$$

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## Joint Probabilities for Multiple Events

Example: Spontaneous Parametric Downconversion

$\square$

## Multipartite Systems and Tensor Product

## Multiple Degrees of Freedom

Consider a physical system with many degrees of freedom (e. g. many particles.)

Pure states of the $t^{\text {th }}$ subsystem is described by a vector in a Hilbert space $h_{i},|\psi\rangle_{i} \in h_{i}$.

Joint state of whole system is a vector in the tensor product space:

$$
|\Psi\rangle \in \mathcal{H}=h_{1} \otimes h_{2} \otimes \cdots \otimes h_{n}
$$

## Example: Bipartite System of Two Photons

$$
\begin{aligned}
\mathcal{H}=\mathbf{C}_{2} \otimes \mathbf{C}_{2}=\left(\mathbf{C}_{2}\right)^{\otimes 2} \quad \text { Four dimensional } \\
|\psi\rangle_{1}=\left[\begin{array}{c}
\alpha \\
\beta
\end{array}\right],|\phi\rangle_{2}=\left[\begin{array}{l}
\gamma \\
\delta
\end{array}\right] \quad \longrightarrow \quad|\psi\rangle_{1} \otimes|\phi\rangle_{2}=\left[\begin{array}{c}
\alpha \\
\beta
\end{array}\right] \otimes\left[\begin{array}{c}
\gamma \\
\delta
\end{array}\right]=\left[\begin{array}{c}
\alpha \gamma \\
\alpha \delta \\
\beta \gamma \\
\beta \delta
\end{array}\right]
\end{aligned}
$$

## Tensor Product: Formal Structure

## Joint Bipartite Hillbert space: $\mathcal{H}_{\mathcal{A B}}=h_{\mathcal{A}} \otimes h_{\mathcal{B}}$

Orthonormal basis for $h_{\mathcal{A}}$

$$
\left\{\left|e_{i}\right\rangle_{A} \mid i=1, \ldots, d_{A}\right\}
$$

Orthonormal basis for $\hbar_{B}$

$$
\left\{\left|f_{j}\right\rangle_{B} \mid j=1, \ldots, d_{B}\right\}
$$

Orthonormal "product basis" for joint space

$$
\left\{\left|E_{i, j}\right\rangle_{A B}=\left|e_{i}\right\rangle_{A} \otimes\left|f_{j}\right\rangle_{B}=\left|e_{i}\right\rangle\left|f_{j}\right\rangle=\left|e_{i}, f_{j}\right\rangle=|i j\rangle\right\}
$$

$\underline{\text { General state vector: }}|\Psi\rangle_{A B}=\sum_{i j} c_{i j}\left|e_{i}\right\rangle_{A} \otimes\left|f_{j}\right\rangle_{B}$

$$
c_{i j}=\left.\left\langle\left.\left\langle\left. e_{i}\right|_{A} \otimes\left\langle\left. f_{j}\right|_{B} \mid \Psi \Psi\right\rangle_{A B}=\langle i j \mid \Psi\rangle_{A B} \quad p(i \text { and } j)=\right|\langle i j \mid \Psi\rangle_{A B}\right|^{2}=\right| c_{i j}\right|^{2}
$$

## Uncorrelated Probabilities

Consider a "product state" in the joint Hilbert Space $\mathcal{H}_{\mathrm{AB}}$

$$
|\Psi\rangle_{A B}=|\psi\rangle_{A} \otimes|\phi\rangle_{B}
$$

Joint Probability of Measurement

$$
\begin{aligned}
P(A=i \text { and } B=j) & =\mid\left(\left.\left\langle\left. e_{i}\right|_{A} \otimes\left\langle\left. f_{j}\right|_{B}\right) \mid \Psi\right\rangle_{A B}\right|^{2}=\left|\left\langle e_{i} \mid \psi\right\rangle_{A}\right|^{2}\left|\left\langle f_{j} \mid \phi\right\rangle_{B}\right|^{2}\right. \\
& =P(A=i) P(B=j)
\end{aligned}
$$

Product state $\Leftrightarrow$ Statistically Uncorrelated Events "Separable State"

## Entangled States

"Quantum Correlated" events = Superposition of joint processes.

Feynman: Add probability amplitudes for indistinguishable processes.

"Type II" Downconversion: Signal and Idler have opposite polarization. But which? Process does not distinguish them --> superposition.

$$
|\Psi\rangle_{s i}=\frac{1}{\sqrt{2}}\left(|H\rangle_{s} \otimes|V\rangle_{i}-|V\rangle_{s} \otimes|H\rangle_{i}\right) \neq|\psi\rangle_{s} \otimes|\phi\rangle_{i}
$$

## Entanglement and correlated collapse

Suppose a measurement of the signal photon's polarization is made in the $\mathrm{H}-\mathrm{V}$ basis and the result " $H$ " is found.

What is the post-measurement state?

$$
|\Psi\rangle_{s i} \Rightarrow|H\rangle_{s}\left\langle H_{s} \mid \Psi\right\rangle_{s i}=\frac{1}{\sqrt{2}}(|H\rangle_{s} \underbrace{\left\langle H_{s} \mid H\right\rangle_{s}}_{1} \otimes|V\rangle_{i}-|H\rangle_{s} \underbrace{\left\langle H_{s} \mid V\right\rangle_{s}}_{0} \otimes|H\rangle_{i}) \Rightarrow|H\rangle_{s} \otimes|V\rangle_{i}
$$

The state of the idler photon "collapses" due to measurement of the signal.

What Alice knows about the Bob's photon is effected by her measurement because she knows the photons are correlated.

## Classical Correlation



## Classical Correlation



## Classical Correlation



Note: Alice and Bob's results are random, but perfectly correlated.

## "Singlet": Anticorrelated in any basis



## "Singlet": Anticorrelated in any basis



## "Singlet": Anticorrelated in any basis

## Proof:

Entangled state of joint system

$$
|\Psi\rangle_{s i}=\frac{1}{\sqrt{2}}\left(|H\rangle_{s} \otimes|V\rangle_{i}-|V\rangle_{s} \otimes|H\rangle_{i}\right)
$$

Measure signal photon

$$
\begin{aligned}
& \hat{\sigma}_{\phi} \\
& \text { eigenvectors }\left|+1_{\phi}\right\rangle=\left[\begin{array}{c}
\cos \phi \\
\sin \phi
\end{array}\right] \quad\left|-1_{\phi}\right\rangle=\left[\begin{array}{c}
-\sin \phi \\
\cos \phi
\end{array}\right] \\
& |\Psi\rangle_{s i} \Rightarrow\left|1_{\phi}\right\rangle_{s}\left\langle 1_{\phi} \mid \Psi\right\rangle_{s i}=\frac{1}{\sqrt{2}}(\left|1_{\phi}\right\rangle_{s} \underbrace{\left\langle_{\phi} \mid H\right\rangle_{s}}_{\cos \phi} \otimes|V\rangle_{i}-\left|1_{\phi}\right\rangle_{s} \underbrace{\hat{1}_{\phi}|V\rangle_{s}}_{\sin \phi} \otimes|H\rangle_{i}) \\
& \Rightarrow\left|1_{\phi}\right\rangle_{s} \otimes\left(\cos \phi|V\rangle_{i}-\sin \phi|H\rangle_{i}\right)=\left|1_{\phi}\right\rangle_{s} \otimes\left|-1_{\phi}\right\rangle_{i}
\end{aligned}
$$

If signal photon is found linear along $\phi$, idler is found in the orthogonal polarization.

## Classical Correlation



## Classical Correlation



## Classical Correlation



## Classical Correlation



## Classical Correlation



## Classical Correlation



Results are uncorrelated.

## EPR Paradox

Can Quantum-Mechanical Description of Physical Reality Be Considered Complete? A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. 47, 777 (1935)

If, without in any way disturbing a system, we can predict with certainty (i.e., with probability equal to unity) the value of a physical quantity, then there exists an element of physical reality corresponding to this physics quantity


EPR argue that, by their definition of "realistic properties", quantum mechanics "incomplete" as it cannot give definite predictions of measurement results that have some definite value (hidden variables).
$\square$

## The EPR Argument

## (Version due to Bohm, 1951)

- Consider entangled state, $|\Psi\rangle_{A B}=\frac{1}{\sqrt{2}}\left(|H\rangle_{A} \otimes|V\rangle_{B}-|V\rangle_{A} \otimes|H\rangle_{B}\right)$
-If Alice were to measure $\hat{Z}$ on her photon, she can, without in any way effecting Bob, determine whether he will find H or V should he perform a $Z$ measurement. $\Longleftrightarrow$ Bob's value of $Z$ is an "element of reality".
-If Alice were to measure $\hat{X}$ on her photon, she can, without in any way effecting Bob, determine whether he will find $\mathrm{D}_{+}$or $\mathrm{D}_{\text {. }}$ should he perform a $X$ measurement. $\Longleftrightarrow$ Bob's values of $X$ is an "element of reality".
- Quantum mechanical states cannot give a simultaneous definite value of both $Z$ and $X$ since these operators don't commute.

Quantum mechanical sates are not a "complete" description of the physical world and can be completed by some "hidden variables".

## John Bell: <br> Putting Hidden Variables to the Test



Bell took EPR seriously, 30 years after they published their original paper and asked when the EPR assumption had any measurable significance. Amazingly...YES!

Bell's Inequality. J.S. Bell, Physics 1195 (1964).

## Formal Statement of EPR

Consider measurement of, $\hat{\sigma}_{\phi}$ linear polarization at angle $\phi$.
According to EPR, the value that Alice measures is a function of her polarizer setting $\phi_{\mathrm{a}}$ and the "realistic hidden variables" $\lambda$. Similarly for Bob, and his polarizer setting $\phi_{b}$.

$$
\text { Measured values: } A\left(\phi_{a}, \lambda\right)= \pm 1 \quad B\left(\phi_{b}, \lambda\right)= \pm 1
$$

The crucial assumption is that $A$ is not a function of $\phi_{b}$, and $B$ is not a function of $\phi_{\mathrm{a}}$. "Local hidden variable theory".

These functions show produce the same statistics as quantum mechanics for some suitable distribution of $\lambda$. They should reproduce the quantum mechanical results.

$$
\left\langle\hat{\sigma}_{\phi_{a}}\right\rangle=\int P(\lambda) A\left(\phi_{a}, \lambda\right)
$$

## Hidden Variable Model for Single Photon Linear Polarization Measurement


$\lambda$ is a unknown unit vector with probability uniformly distributed in the blue $1 / 4$-wedge.

Deterministic binary choice, given $\lambda: \quad+1$ if $\lambda$ is in pink $1 / 4$-wedge.
-1 if $\lambda$ is not.

$$
p_{+1}=\frac{\alpha(\theta, \phi)}{\pi / 2} \quad p_{-1}=\frac{\pi / 2-\alpha(\theta, \phi)}{\pi / 2}
$$

Choose: $\alpha(\theta, \phi)=\frac{\pi}{2} \cos ^{2}(\phi-\theta)$

## Correlation functions

Averages of joint observables

$$
\begin{aligned}
\left\langle\hat{\sigma}_{\phi_{a}} \otimes \hat{\sigma}_{\phi_{b}}\right\rangle & =E\left(\phi_{a}, \phi_{b}\right) \quad \text { (Expectation value) } \\
& =\left\langle\Psi_{A B}\right| \hat{\sigma}_{\phi_{a}} \otimes \hat{\sigma}_{\phi_{b}}\left|\Psi_{A B}\right\rangle \\
& =\int d \lambda P_{A B}(\lambda) A\left(\phi_{a}, \lambda\right) B\left(\phi_{b}, \lambda\right)
\end{aligned}
$$

For quantum mechanical singlet state

$$
\begin{gathered}
\left|\Psi_{A B}\right\rangle=\frac{1}{\sqrt{2}}\left(|H\rangle_{A}|V\rangle_{B}-|V\rangle_{A}|H\rangle_{B}\right) \\
\left\langle\hat{\sigma}_{\phi_{a}} \otimes \hat{\sigma}_{\phi_{b}}\right\rangle=-\cos \left[2\left(\phi_{a}-\phi_{b}\right)\right]
\end{gathered}
$$

Can the local hidden variable theory mimic the QM prediction?

## Measuring Correlation Functions

Coincidence counting

$$
\left\langle\hat{\sigma}_{\phi_{a}} \otimes \hat{\sigma}_{\phi_{b}}\right\rangle=C\left(1_{\phi_{a}}, 1_{\phi_{b}}\right)+C\left(-1_{\phi_{a}},-1_{\phi_{b}}\right)-C\left(1_{\phi_{a}},-1_{\phi_{b}}\right)-C\left(-1_{\phi_{a}}, 1_{\phi_{b}}\right)
$$



## Bell/CHSH Inequality

(Clauser-Horne-Shimony-Holt version, Phys. Rev. Lett. 23880 (1969))
Alice and Bob can choose to measure in one of two local basis

$$
\left\{\phi_{a}, \phi_{a^{\prime}}\right\}
$$

$$
S_{\operatorname{mpx}_{3}}\left\{\phi_{b}, \phi_{b^{\prime}}\right\}
$$

Consider the following correlation joint observable

$$
\begin{aligned}
& \hat{S}=\hat{\sigma}_{a} \otimes\left(\hat{\sigma}_{b^{\prime}}-\hat{\sigma}_{b}\right)+\hat{\sigma}_{a^{\prime}} \otimes\left(\hat{\sigma}_{b^{\prime}}+\hat{\sigma}_{b}\right) \\
& \langle\hat{S}\rangle=E\left(\phi_{a}, \phi_{b^{\prime}}\right)-E\left(\phi_{a}, \phi_{b}\right)+E\left(\phi_{a^{\prime}}, \phi_{b^{\prime}}\right)+E\left(\phi_{a^{\prime}}, \phi_{b}\right)
\end{aligned}
$$

The value of $S_{\lambda}$ in a local hidden variable model

$$
\begin{gathered}
S_{\lambda}=A\left(\phi_{a}, \lambda\right)\left[B\left(\phi_{b^{\prime}}, \lambda\right)-B\left(\phi_{b}, \lambda\right)\right]+A\left(\phi_{a^{\prime}}, \lambda\right)\left[B\left(\phi_{b^{\prime}}, \lambda\right)+B\left(\phi_{b}, \lambda\right)\right] \\
S_{\lambda}= \pm 2 \\
\square-2 \leq\langle S\rangle_{L H V} \leq 2
\end{gathered}
$$

## QM Violates LHV

In singlet state:
$\langle\hat{S}\rangle_{Q M}=-\cos \left[2\left(\phi_{a}-\phi_{b^{\prime}}\right)\right]+\cos \left[2\left(\phi_{a}-\phi_{b}\right)\right]-\cos \left[2\left(\phi_{a^{\prime}}-\phi_{b^{\prime}}\right)\right]-\cos \left[2\left(\phi_{a^{\prime}}-\phi_{b}\right)\right]$
Consider case where the polarizer directions are equally spaced as follows:



$$
\langle\hat{S}\rangle_{Q M}=3 \cos 2 \theta-\cos (6 \theta)
$$

Quantum correlations stronger than classical correlations.

Max violation:

- Alice measures $\hat{X}$ or $\hat{Z}$
- Bob measures $\frac{\hat{X}+\hat{Z}}{\sqrt{2}}$ or $\frac{\hat{X}-\hat{Z}}{\sqrt{2}}$


## Experimental Verification

- First definitive test:

Aspect et al. 1982, Phys. Rev. Lett. 491804.


Atomic cascade
6 standard deviations averaged for hours.

- Modern tests:
P.G. Kwiat et al. 1995, 754337


Bright source of 100 standard entangled photons deviations in $<5 \mathrm{~min}$.

## Implications

- Quantum mechanics cannot be described by a local realistic theory.
- Nonlocal hidden variables?
- No "realistic properties" of observed quantities.
- Entanglement CANNOT be used to communicate faster than the speed of light.
- Alice and Bobs results are random but correlated.
- Entanglement as a resource.
- Quantum correlations are special and destroyed by "eavesdropper".
- Communication tasks aided by shared entanglement:
- Quantum dense coding.
- Teleportation.
- Distributed computation (e.g. appointment problem).

