Decision Diagrams
– Theoretical and Practical Aspects –

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$$\mathrm{f}: \quad \mathrm{B}^{n} \rightarrow \mathrm{B}^{m}$$

MULT

Let $a = a_{n-1} \ldots a_1 a_0$ and $b = b_{n-1} \ldots b_1 b_0$ be n-bit numbers

$$\mathrm{MULT}_i(a, b) := \text{ i-th bit of the product of a and b}$$

$\oplus$-$\boldsymbol{cl}_{n,3}$, 1-$\boldsymbol{cl}_{n,3}$

Given n points in the plane,

set x of undirected edges

$\oplus$-$\boldsymbol{cl}_{n,3}(x) = 1 \quad :\Longleftrightarrow \quad x$ defines an odd number of 3-cliques

1-$\boldsymbol{cl}_{n,3}(x) = 1 \quad :\Longleftrightarrow \quad x$ conists of exactly 3 edges, which form a 3-clique

$$f : \quad B^n \to B$$

data structure for Boolean functions

Requests:

- small representation

- efficient manipulation:
  - evaluation, satisfiability, equivalence
  - operations $f_1 \circ f_2$
  - ...

well-known:	function tables circuits formula
here:	**Decision Diagrams**

Overview

I. Fundamentals on DDs

II. τ-operator

III. Gaps, lower bounds

IV. FDD-package, experiments

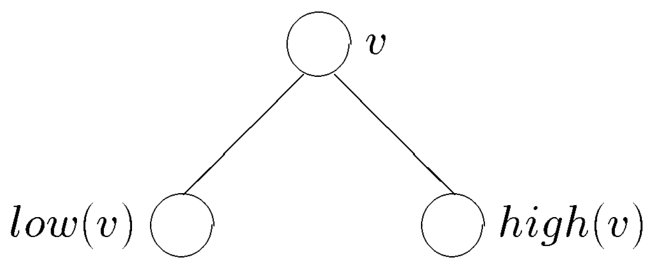
I. Fundamentals on DDs

Decision Diagram (DD)

directed, acyclic graph $G = (V, E)$

with:

- exactly one root
- nonterminal nodes



$$label(v) \in \{x_1, x_2, \ldots, x_n\}$$

- terminal nodes

$$\boxed{0}$$

$$label(v) \in \{0, 1\}$$

Interpretation

DD G $\quad\rightsquigarrow\quad$ **Binary Decision Diagram (BDD)**

$$f_G^{BDD}$$

$\rightsquigarrow\quad$ **Functional Decision Diagram (FDD)**

$$f_G^{FDD}$$

1. G consists of one node with *label* 0 (1):

$$f_G^{BDD} \;=\; f_G^{FDD} \;=\; 0 \;\;(1)$$

2. G has root v with *label* x_i:

$$\begin{aligned} f_G^{BDD} &= \bar{x}_i\, f_{G_{low(v)}}^{BDD} \;+\; x_i\, f_{G_{high(v)}}^{BDD} \\ f_G^{FDD} &= f_{G_{low(v)}}^{FDD} \;\oplus\; x_i\, f_{G_{high(v)}}^{FDD} \end{aligned}$$

$$
\begin{aligned}
f &= \overline{x}_i f_{\overline{x}_i} + x_i f_{x_i} \\
f &= f_{\overline{x}_i} \oplus x_i (f_{\overline{x}_i} \oplus f_{x_i})
\end{aligned}
$$

Lemma

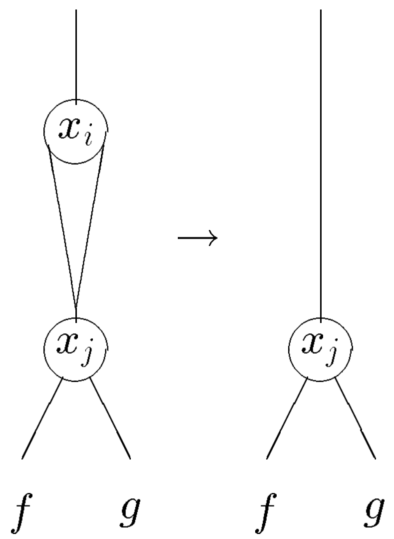
- There exist BDDs, FDDs for any $f : \mathbf{B}^n \to \mathbf{B}$.
- Evaluation of BDDs is possible in time $O(\text{depth})$.
- Evaluation of FDDs is possible in time $O(\text{size})$.

Example:

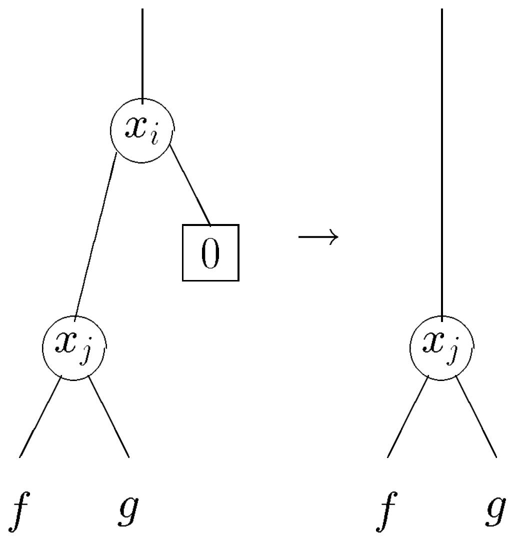
Reduction

Type I: Isomorphic Subgraphs

Typ S:



Typ D:



For $M \subset \{I, S, D\}$:

- *(t_M)-equivalence* of DDs G_1 and G_2
- *(t_M)-reduction* of G

Lemma

- $f_{G_1}^{\mathrm{BDD}} = f_{G_2}^{\mathrm{BDD}}$ for $(t_{\{I,S\}})$-equivalent DDs.
- $f_{G_1}^{\mathrm{FDD}} = f_{G_2}^{\mathrm{FDD}}$ for $(t_{\{I,D\}})$-equivalent DDs.
- Reductions can be computed in linear time.

Structural Restrictions

- DD ordered:

  variables in same order on any path $\rightsquigarrow$ OBDDs, OFDDs

- DD free:

  each variable at most once on each path $\rightsquigarrow$ FBDDs, FFDDs

- DD complete:

  each variable exactly once on each path $\rightsquigarrow$ CBDDs, CFDDs

Lemma

- $(t_{\{I,S\}})$-reduced OBDD for f and given ordering π is unique.
- $(t_{\{I,D\}})$-reduced OFDD for f and given ordering π is unique.

Theorem 1

Equivalence and satisfiability for OBDDs and OFDDs with fixed ordering π can be efficiently decided.

BP, BPk, BP$_{polsize}$

Lee '59, Akers '78

Zák '84, Ajtai ... '86,

Wegener '87, Meinel '89, ...

OBDD, BDD-packages

Bryant '86

Bryant, Brace, Rudell '90

Somenzi '91, ...

B.'92, Krieger, B., Sinković '93

OFDD

Kebschull, Rosenstiel, Schubert '92, '93

II. τ-Operator

What is the relation between

$$f_G^{\mathrm{BDD}} \quad \text{and} \quad f_G^{\mathrm{FDD}} \;?$$

$$f_G^{\mathrm{FDD}}(a) \;=\; \bigoplus_{b \leq a} f_G^{\mathrm{BDD}}(b)$$

Define τ for $f \in \mathbf{B}_n$ by

$$\tau(f)(x) = \bigoplus_{y \leq x} f(y)$$

(see also Razborov '86, Beßlich, Trachtenberg '92)

Theorem 2

Let G be a complete DD. Then:

$$\begin{aligned} f_G^{\mathrm{FDD}} &= \tau\left(f_G^{\mathrm{BDD}}\right) \\ f_G^{\mathrm{BDD}} &= \tau\left(f_G^{\mathrm{FDD}}\right) \end{aligned}$$

For OBDDs, OFDDs, FBDDs, FFDDs ?

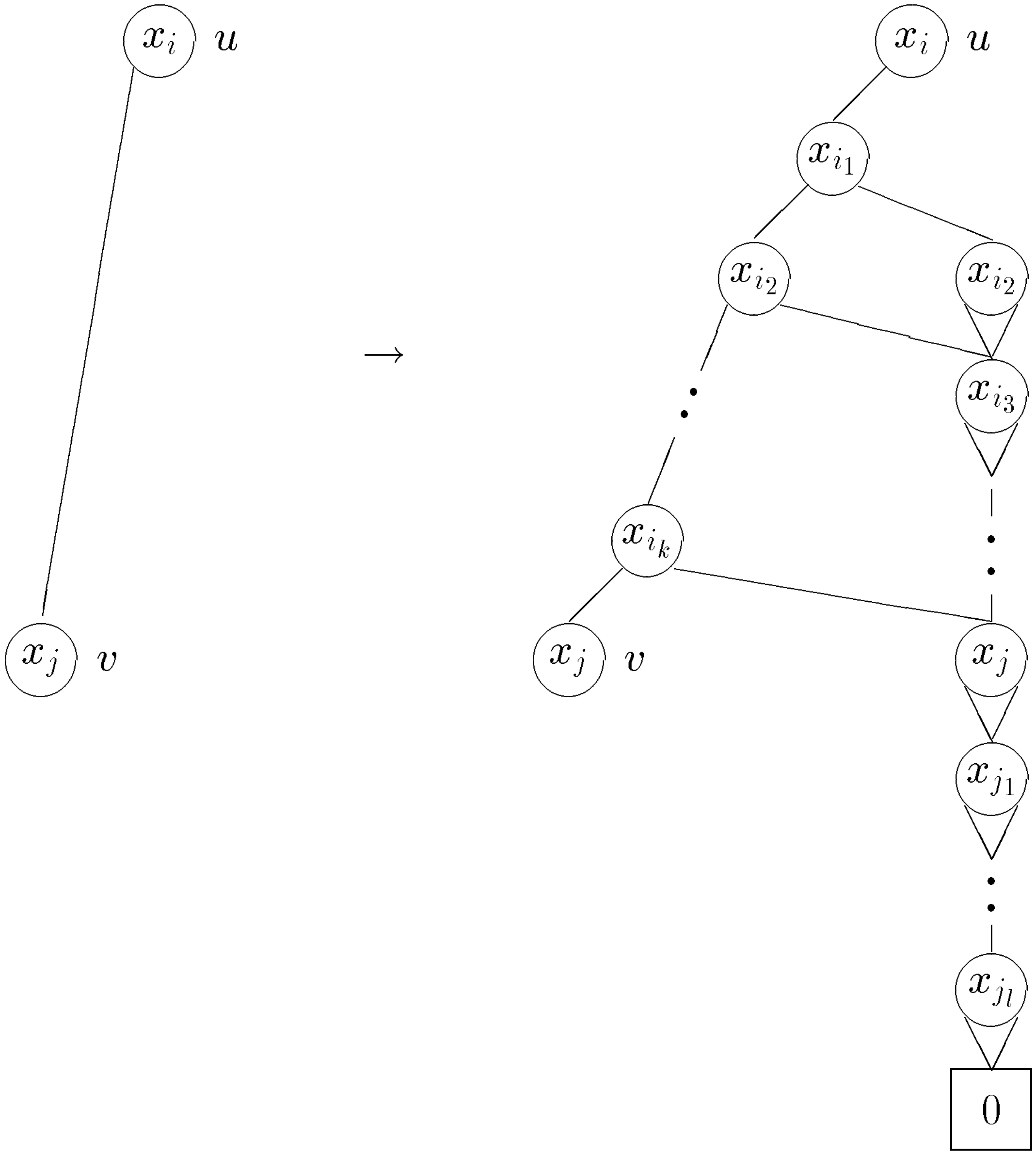
Theorem 3

$$\begin{aligned}
\Omega(1/n) \cdot \text{OBDD-SIZE}_\pi(\tau(f)) \quad &\leq \quad \text{OFDD-SIZE}_\pi(f) \quad \leq \quad O(n) \cdot \text{OBDD-SIZE}_\pi(\tau(f)) \\
\Omega(1/n) \cdot \text{OBDD-SIZE}(\tau(f)) \quad &\leq \quad \text{OFDD-SIZE}(f) \quad \leq \quad O(n) \cdot \text{OBDD-SIZE}(\tau(f)) \\
\Omega(1/n) \cdot \text{FBDD-SIZE}(\tau(f)) \quad &\leq \quad \text{FFDD-SIZE}(f) \quad \leq \quad O(n) \cdot \text{FBDD-SIZE}(\tau(f))
\end{aligned}$$

Sketch of Proof:

- Make each free (ordered) DD complete. Show, that the size increases at most by a factor of $O(n)$.

- Apply Theorem 2.



x_i
u
x_j
v
$\to$
x_i
u
x_{i_1}
x_{i_2}
x_{i_2}
x_{i_3}
x_{i_k}
x_j
v
x_j
x_{j_1}
x_{j_l}
0

First Consequences

- variable ordering is important

  e.g. $f = \tau(x_1x_2 + .. + x_{2n-1}x_{2n})$

- functions with polynomial FFDDs,
  but exponential OFDDs

  e.g. $f = \tau\,(HWB)$

- complete OFDDs of size

  $$n(n-1)/2$$

  for symmetric functions

III. Gaps, Lower Bounds

Consider $n(n-1)/2$ variables x_{ij} $(1 \leq i < j \leq n)$.

$$\oplus\text{-}cl_{n,3}(x) = 1 \quad :\Longleftrightarrow \quad G(x) \text{ contains an odd number of 3-cliques}$$

$$1\text{-}cl_{n,3}(x) = 1 \quad :\Longleftrightarrow \quad G(x) \text{ contains exactly one 3-clique and } n-3 \text{ isolated points}$$

Theorem 4 [Exponential gaps]

For each variable ordering:

$$\begin{aligned}
\text{OBDD-SIZE}_\pi\,(1\text{-}\boldsymbol{cl}_{n,3}) &= O(n^5) \quad \text{but} && (1)\\
\text{FFDD-SIZE}\,(1\text{-}\boldsymbol{cl}_{n,3}) &= 2^{\Omega(n)} && (2)
\end{aligned}$$

$$\begin{aligned}
\text{OFDD-SIZE}_\pi\,(\oplus\text{-}\boldsymbol{cl}_{n,3}) &= O(n^3) \quad \text{but} && (3)\\
\text{FBDD-SIZE}\,(\oplus\text{-}\boldsymbol{cl}_{n,3}) &= 2^{\Omega(n)} && (4)
\end{aligned}$$

Sketch of proof:

Lower bounds:

$$\oplus\text{-}\boldsymbol{cl}_{n,3}(x) \;=\; \bigoplus_{y \le x} 1\text{-}\boldsymbol{cl}_{n,3}(y) \;=\; \tau(1\text{-}\boldsymbol{cl}_{n,3})$$

Ajtai et al. '86 + Theorem 3

Upper bounds:

Consider complete tree

and estimate size of the reductions.

Theorem 5 [Exponential lower bounds]

$$\begin{aligned} \text{OFDD-SIZE}\,(HWB\,) &= 2^{\Omega(n)} \\ \text{OFDD-SIZE}\,(\,\text{MULT}^n_{n-1}\,) &= 2^{\Omega(n)} \end{aligned}$$

Principle of the proof:

Prove lower bounds for OBDD-size of

$\tau(HWB\,)$, $\tau(\,\text{MULT}^n_{n-1}\,)$.

IV. FDD-package, experiments

software-package for efficient representation and manipulation of OFDDs

- **Implementational Details**

  shared OFDDs,

  hash-based unique table, computed table,

  level lists for exchange of variables,

  memory management by garbage collection,

  (including dynamic variable reordering)

- **Optimization of representation size**

- **Operations on OFDDs**

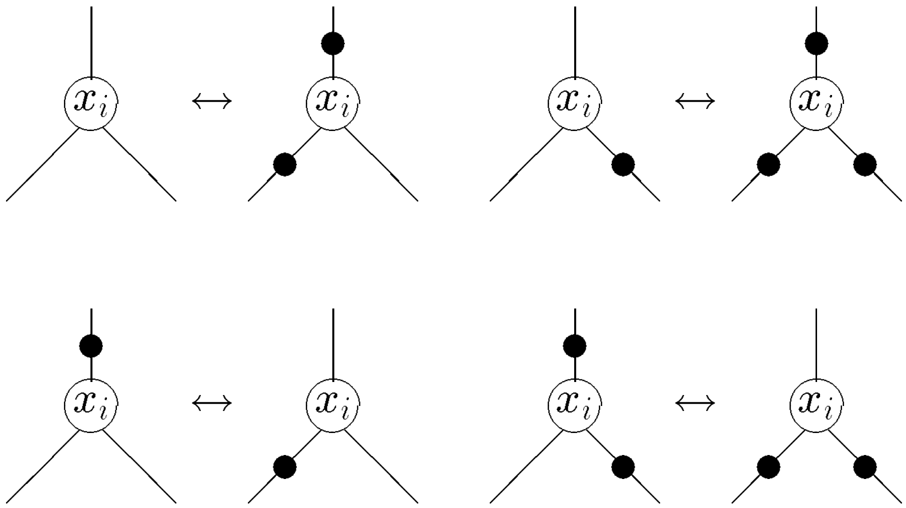
Optimization of representation size

- **$(t_{\{I,D\}})$-reduced OFDDs**

- **complement–edges:**

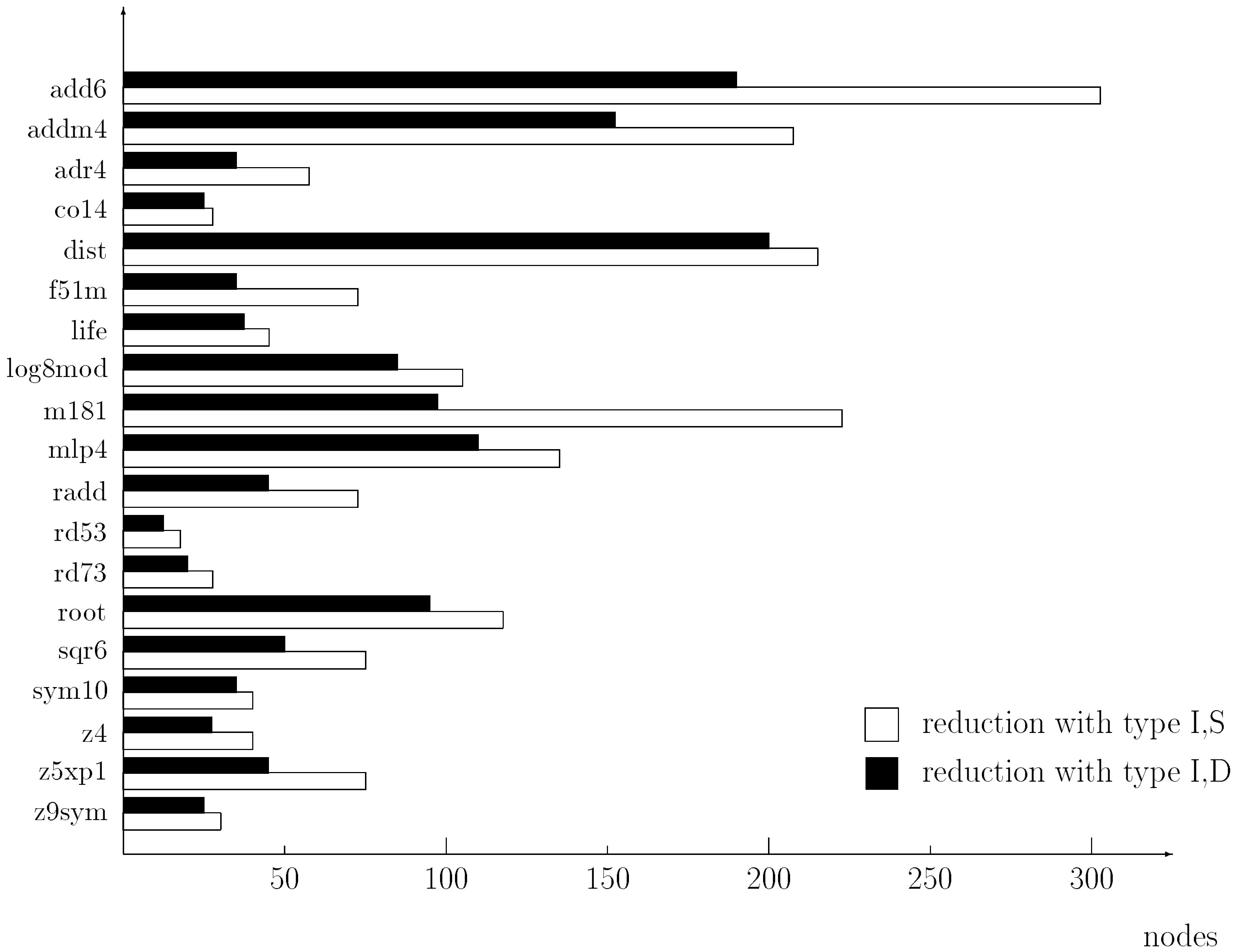
  Use ONE node for f and $\overline{f}$

  problem: uniqueness of representation



complement can be computed in linear time

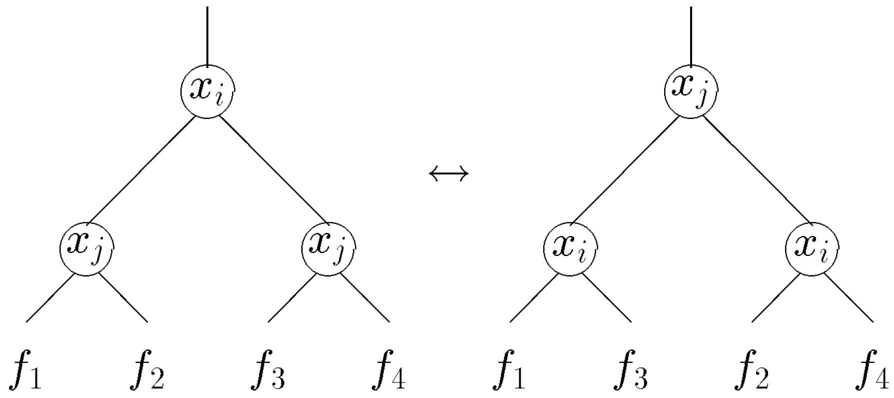
10 % size reduction



add6
addm4
adr4
co14
dist
f51m
life
log8mod
m181
mlp4
radd
rd53
rd73
root
sqr6
sym10
z4
z5xp1
z9sym
reduction with type I,S
reduction with type I,D
50
100
150
200
250
300
nodes

- **variable ordering:**

  exchange of adjacent variables



$\rightsquigarrow$ heuristics [Fujita et al. 91, Rudell 93]

optimal ordering [Ishiura et al. 91]

$name$	$size_{min}$	$m = 2$	$m = 3$	$m = 4$	$sift$
add6	23	58	59	57	45
addm4	132	133	132	132	132
adr4	15	15	15	15	15
dist	159	172	172	171	159
f51m	35	35	35	35	35
life	24	38	25	25	25
mlp4	107	107	107	107	107
radd	15	27	27	26	20
rd53	13	13	13	13	13
rd73	21	21	21	21	21
sqr6	50	50	50	50	50
sym10	36	36	36	36	36
z4	13	13	13	13	13
z5xp1	45	45	45	45	45
z9sym	26	26	26	26	26

OFDD size for dynamic variable ordering

name	*[SKR93]*	*sift*
b12	292	61
cordic	233	41
cps	4922	1293
ex5	901	272
rd53	34	13
rd73	64	21
rd84	86	29
sao2	154	98
t481	54	20
vg2	1912	245
Z5xp1	54	45
Z9sym	40	20

Comparison with [SKR93]

Operations on OFDDs

- **complement**
- **AND-operation:**

$$\begin{aligned} f \cdot g &= (f_{low} \oplus x_i f_{high}) \cdot (g_{low} \oplus x_i g_{high}) \\ &= (f_{low} \cdot g_{low}) \oplus x_i((f_{low} \cdot g_{low}) \oplus (f_1 \cdot g_1)) \\ &= (f_{low} \cdot g_{low}) \oplus x_i((f_{high} \cdot g_{high}) \oplus (f_{low} \cdot g_{high}) \oplus (g_{low} \cdot f_{high})) \end{aligned}$$

but:

Corollary

And-operation may lead to an exponential increase of OFDD-size.

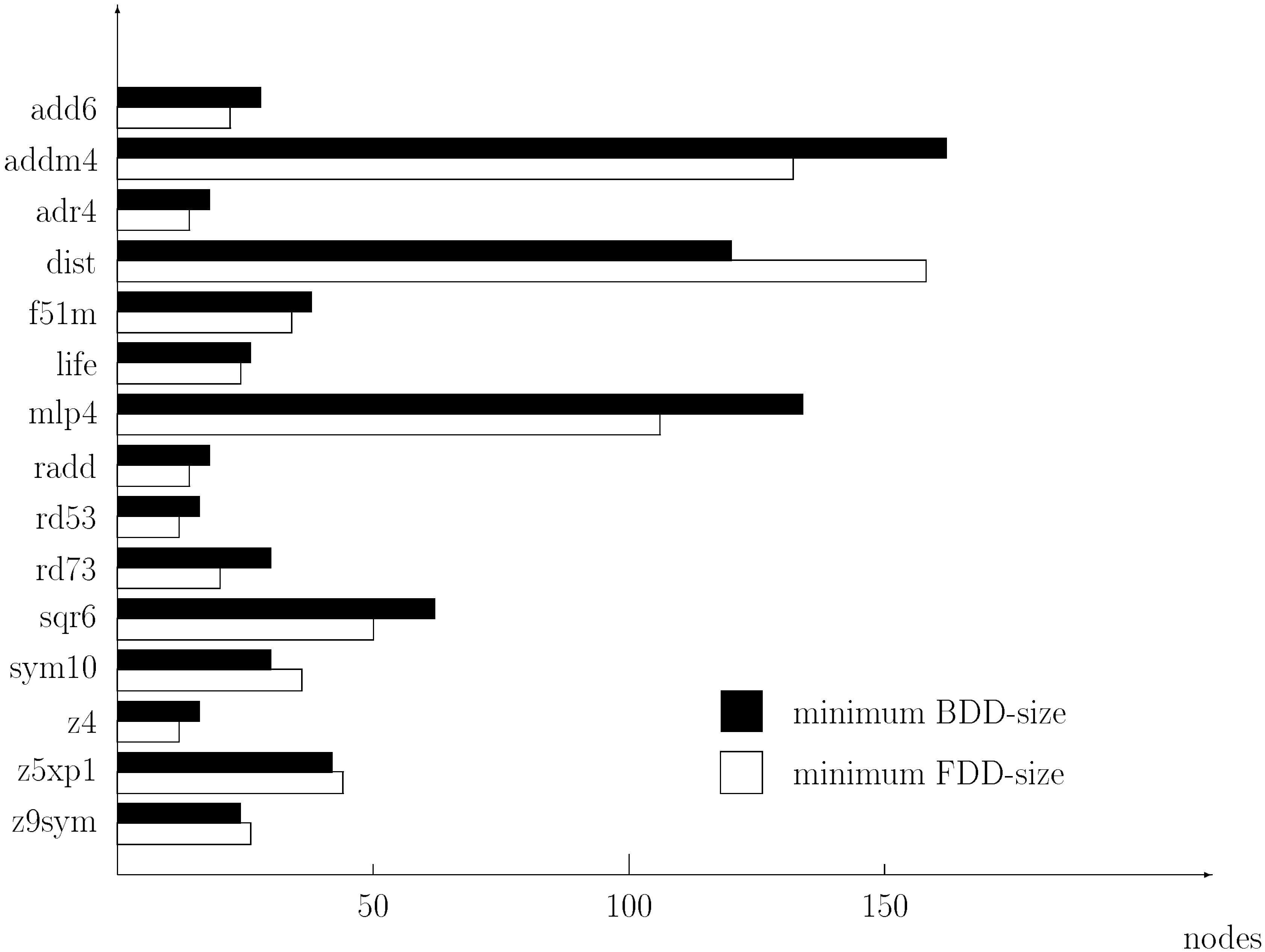
Proof:

$$1\text{-}cl_{n,3} = \oplus\text{-}cl_{n,3} \cdot T^{\leq 3}_{n(n-1)/2}$$

- **EXOR-operation:**

$$
\begin{aligned}
f \oplus g &= (f_{low} \oplus x_i f_{high}) \oplus (g_{low} \oplus x_i g_{high}) \\
&= (f_{low} \oplus g_{low}) \oplus x_i (f_{high} \oplus g_{high})
\end{aligned}
$$

$\rightsquigarrow$ algorithm in time $O(|F| \cdot |G|)$


add6
addm4
adr4
dist
f51m
life
mlp4
radd
rd53
rd73
sqr6
sym10
z4
z5xp1
z9sym
minimum BDD-size
minimum FDD-size
50
100
150
nodes

name	*OBDD*	*OFDD*
s27	9	9
s298	86	71
s344	103	115
s349	103	115
s382	121	122
s1196	631	734
s1423	2725	2044
s1488	385	386
s1494	385	386

OFDD-size for ISCAS benchmarks

Conclusions

- Relation between FDDs and BDDs
- "good" and "bad" functions for OFDDs (OBDDs)
- complexity analysis of manipulation algorithms
- practical experiences

- general FDDs
- larger examples

  [Becker, Drechsler, Theobald 93]
- hybrid DDs

  [Drechsler, Sarabi, Theobald, Becker, Perkowski 93]
- minimal two-level AND/EXOR representations

  [Becker, Drechsler, Theobald 93]
- DD-circuits

  [Becker, Drechsler 93]

name	#FDD-nodes
c432	4479
s1196	734
s1423	2044
dalu	606
des	2666
seq	1252
apex1	2778