# SCHEDULING



Stanford University

# Outline

- The scheduling problem.
- Scheduling without constraints.
- Scheduling under timing constraints.
  - Relative scheduling.
- Scheduling under resource constraints.
  - The ILP model (Integer Linear Programming).
  - Heuristic methods (graph coloring, etc).

Timing constraints versus ***resource constraints***

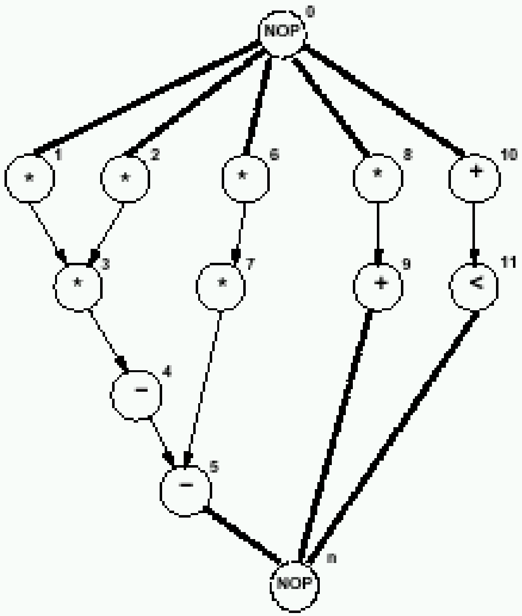
# Scheduling

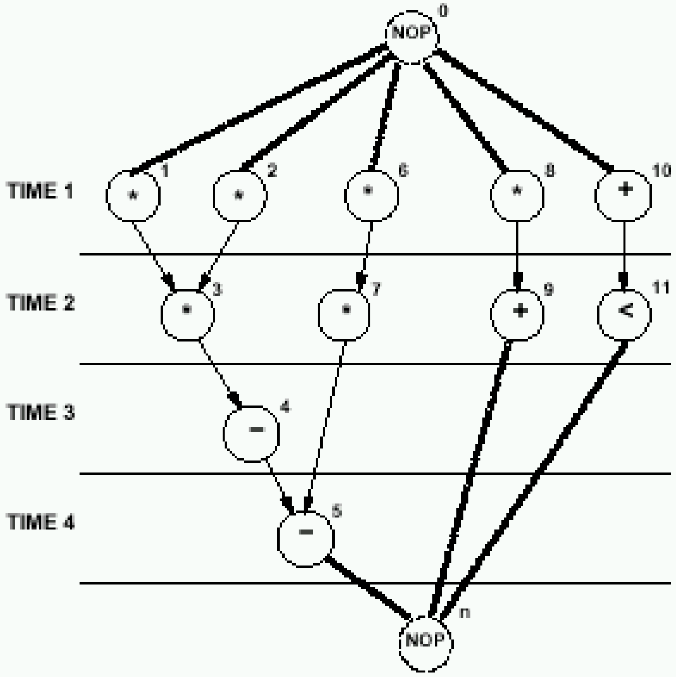
- Circuit model:
  - Sequencing graph.
  - Cycle-time is given.
  - Operation delays expressed in cycles.
- Scheduling:
  - Determine the start times for the operations.
  - Satisfying all the sequencing (timing and resource) constraints.
- Goal:
  - Determine ***area/latency*** trade-off.

**Do you remember what is latency?**

# Example

This is As Soon as Possible Scheduling (ASAP). It can be used as a bound in other methods like ILP or when latency only is important, not area.



# Taxonomy

- Unconstrained scheduling.
- Scheduling with timing constraints:
  - Latency.
  - Detailed timing constraints.
- Scheduling with resource constraints.
- Related problems:
  - Chaining. What is chaining?
  - Synchronization. What is synchronization?
  - Pipeline scheduling.

# Simplest model

- All operations have bounded delays.
- All delays are expressed in numbers of cycles of a single one-phase clock.
  - Cycle-time is given.
- No constraints - no bounds on area.
- ***Goal:***
  - Minimize latency.

# Minimum-latency unconstrained scheduling problem

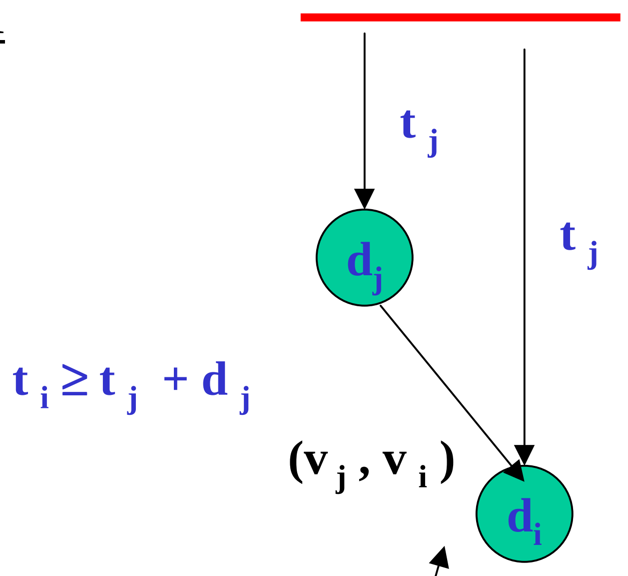
- Given a <u>set of operations</u> **V** with set of corresponding <u>integer delays</u> **D** and a <u>partial order on the operations</u> **E**:
- Find an <u>integer labeling</u> of the <u>operations</u>

  $\varphi : V \rightarrow Z^+$, such that:

  – $t_i = \varphi(v_i)$,

  – $t_i \geq t_j + d_j \; \forall$ i, j such that $(v_j, v_i) \in E$

  – and $t_n$ is ***minimum***.




**Input to $d_i$ must be stable**

# ASAP scheduling algorithm

ASAP ( $G_s(V, E)$){

Schedule $v_0$ by setting $t^S_0 = 1$;

repeat {

Select a vertex $v_i$ whose predecessors are all scheduled;

Schedule $v_i$ by setting $t^S_i = \max_{j:(v_j,v_i)\in E} t^S_j + d_j$ ;

}

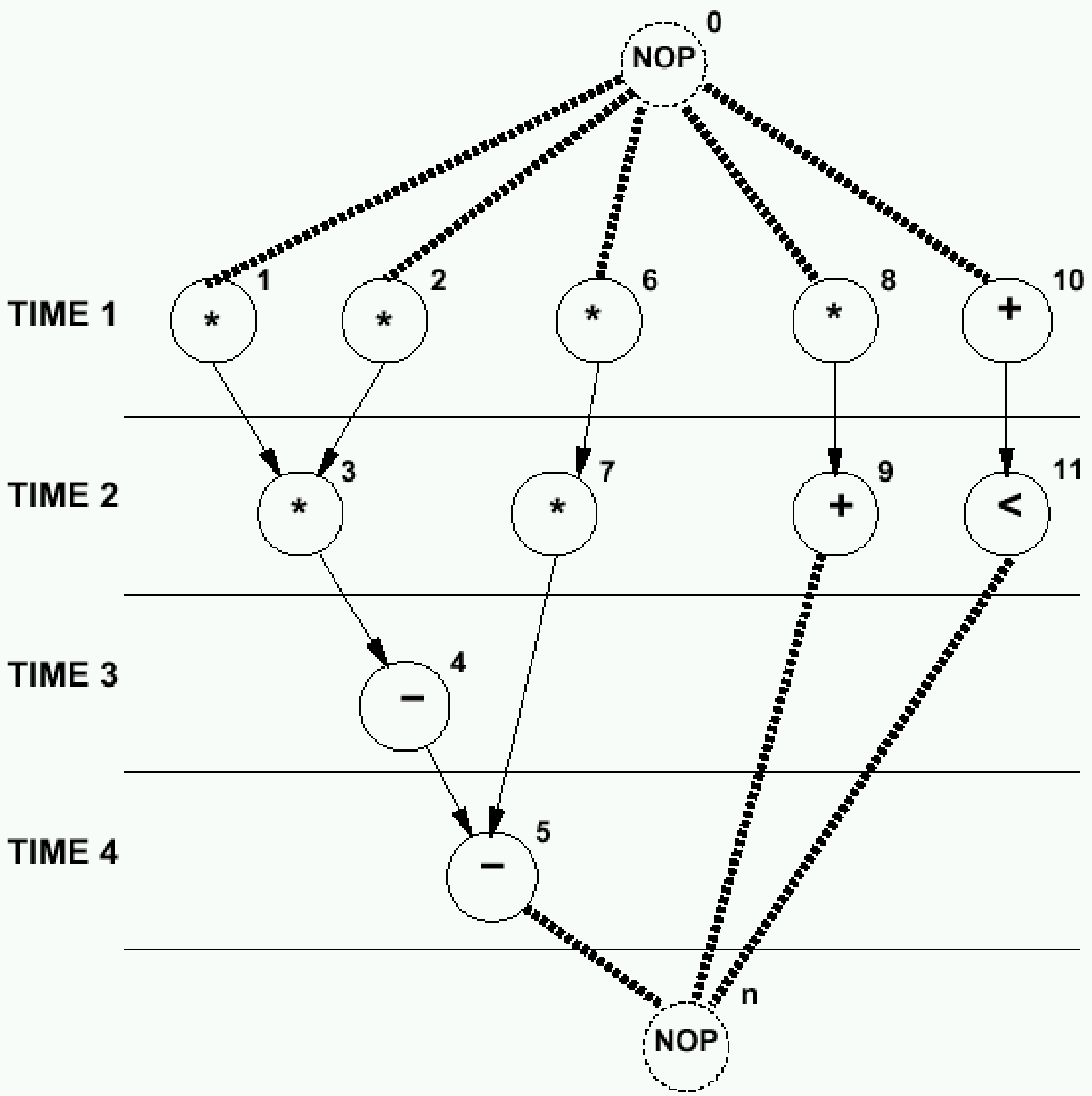
until ($v_n$ is scheduled) ;

return ($t^S$);

}

**Similar to breadth-first search**

# Example - ASAP




- Solution
- Multipliers = 4
- ALUs = 2

**Latency Time=4**

# ALAP scheduling algorithm

$ALAP(\ G_s(V,E), \overline{\lambda})\ \{$

Schedule $v_n$ by setting $t_n^L = \overline{\lambda} + 1$;

**repeat** $\{$

Select vertex $v_i$ whose succ. are all scheduled;

Schedule $v_i$ by setting $t_i^L = \min\limits_{j:(v_i,v_j)\in E} t_j^L - d_i$ ;

$\}$

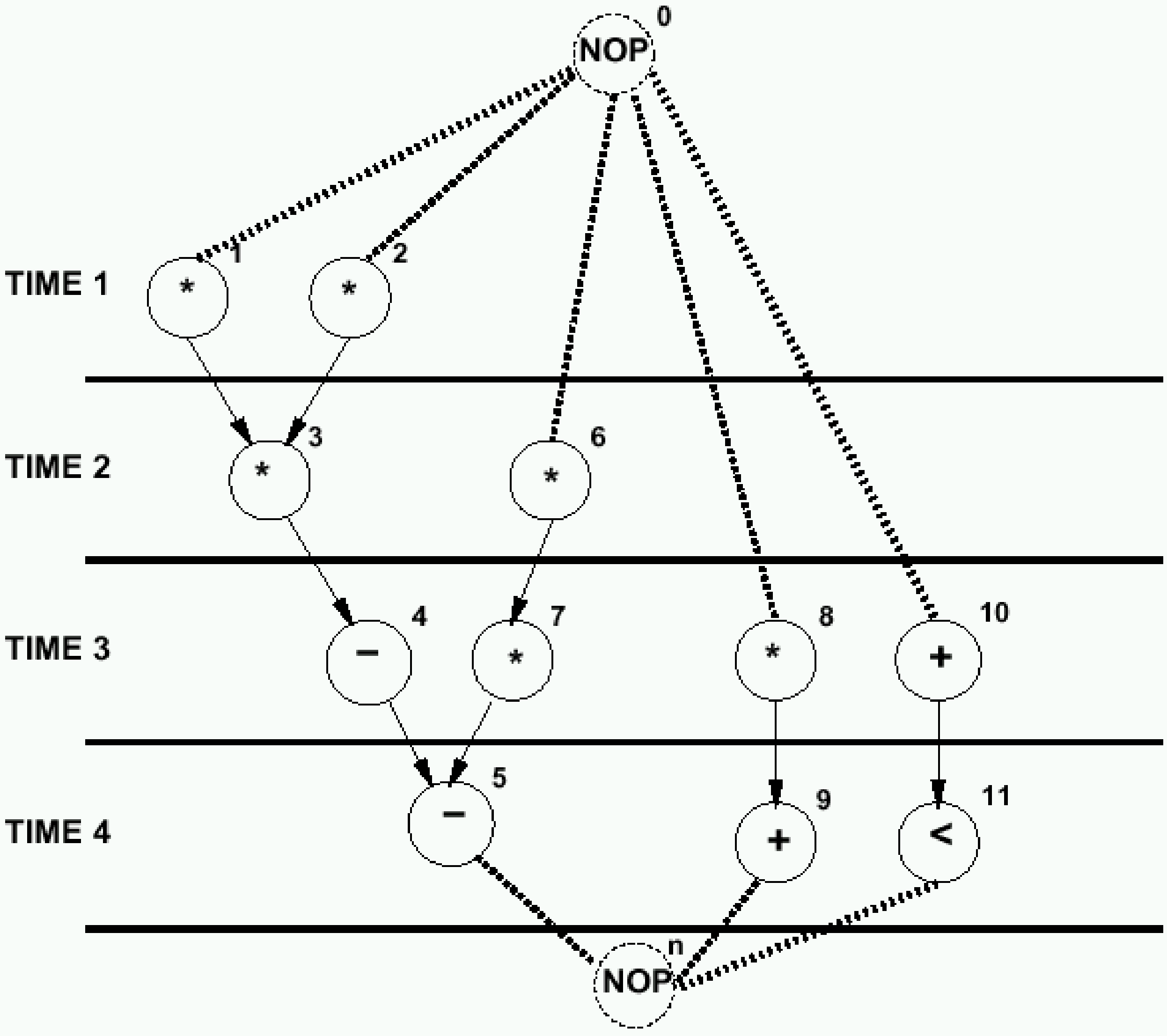
**until** ($v_0$ is scheduled) ;

**return** ($\mathbf{t}^L$);

$\}$

- As Late as Possible - ALAP
- Similar to depth-first search

# Example ALAP




- Solution
- multipliers = 2
- ALUs = 3

**Latency Time=4**

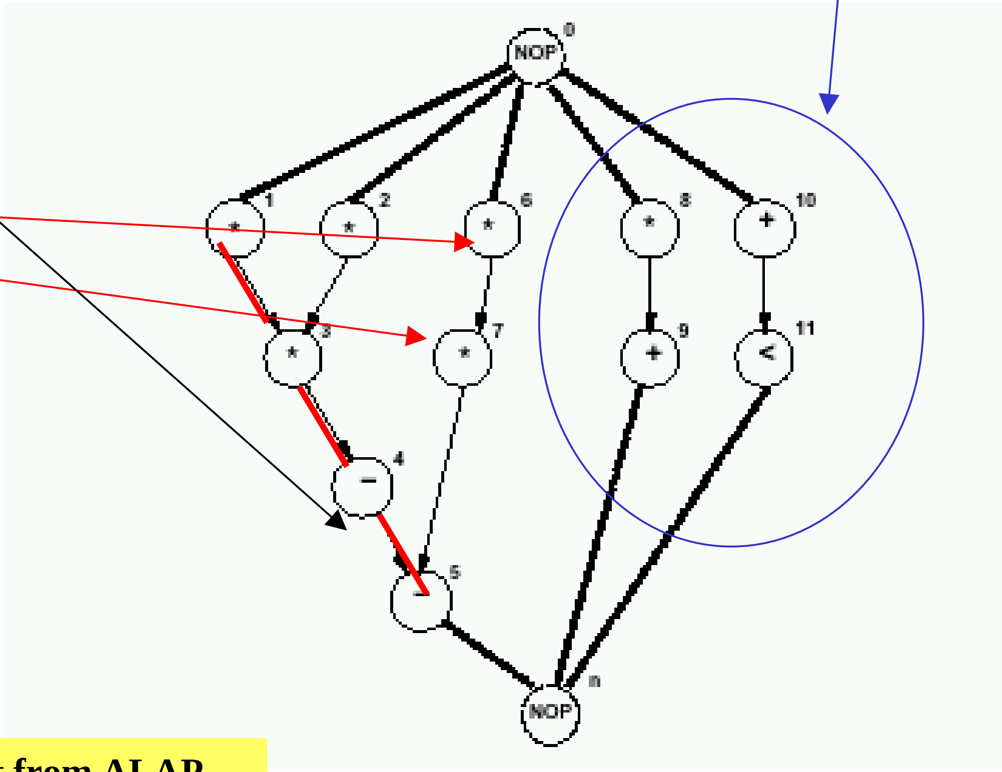
# Remarks

- ALAP solves a latency-constrained problem.
- Latency bound can be set to latency computed by ASAP algorithm. <-- using bounds, also in other approaches
- **Mobility:**
  - Mobility is defined for each operation.
  - Difference between ALAP and ASAP schedule.
- What is **mobility?number of cycles that I can move upwards or downwards the operation**
- Slack on the start time.

# Example of using mobility

- Operations with zero mobility:
  - $\{v_1, v_2, v_3, v_4, v_5\}$.
  - ***They are on the critical path.***
- Operations with mobility one:
  - $\{v_6, v_7\}$.
- Operations with mobility two:
  - $\{v_8, v_9, v_{10}, v_{11}\}$.

mobility two:




**1. Start from ALAP**

**2.Use mobility to improve**

**Last slide for today**

# Scheduling under *detailed timing* constraints

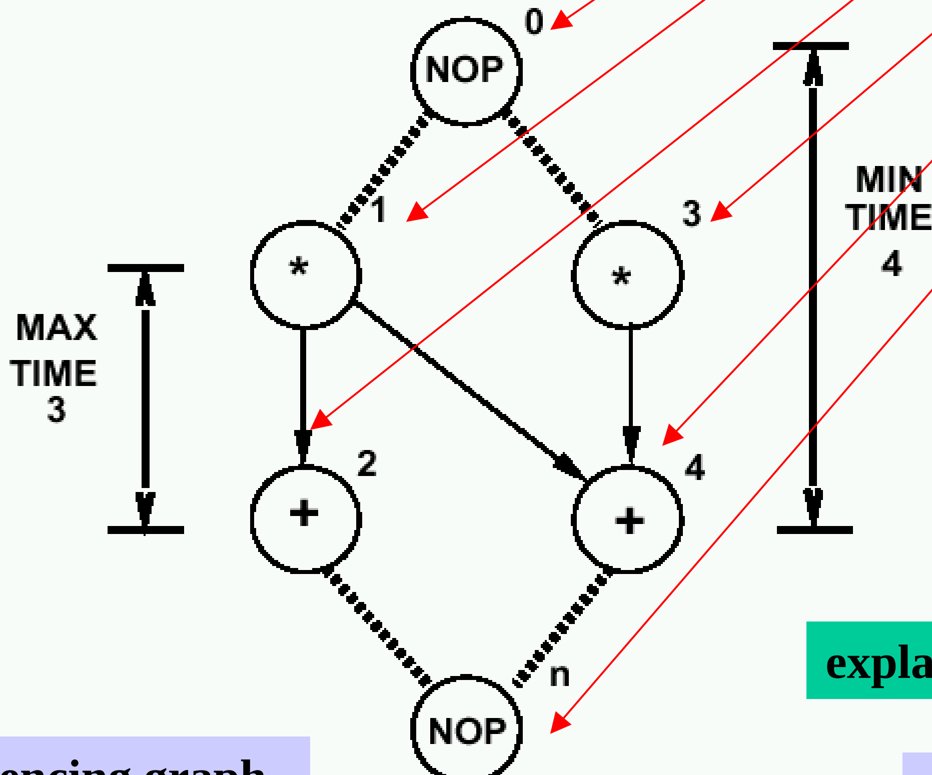
- Motivation:
  - Interface design.
  - Control over operation start time.
- Constraints:
  - Upper/lower bounds on start-time difference of any operation pair.
- Feasibility of a solution.

# Constraint graph model

- Start from a sequencing graph.
- Model delays as weights on edges.
- Add forward edges for *minimum* constraints.
  - Edge $(v_i , v_j)$ with weight $l_{ij} \Rightarrow t_j \geq t_i + l_{ij}$ .
- Add backward edges for *maximum* constraints.
  - Edge $(v_i , v_j)$ with weight:
    - $- u_{ij} \Rightarrow t_j \leq t_i + u_{ij}$
  - because $t_j \leq t_i + u_{ij} \Rightarrow t_i \geq t_j - u_{ij}$

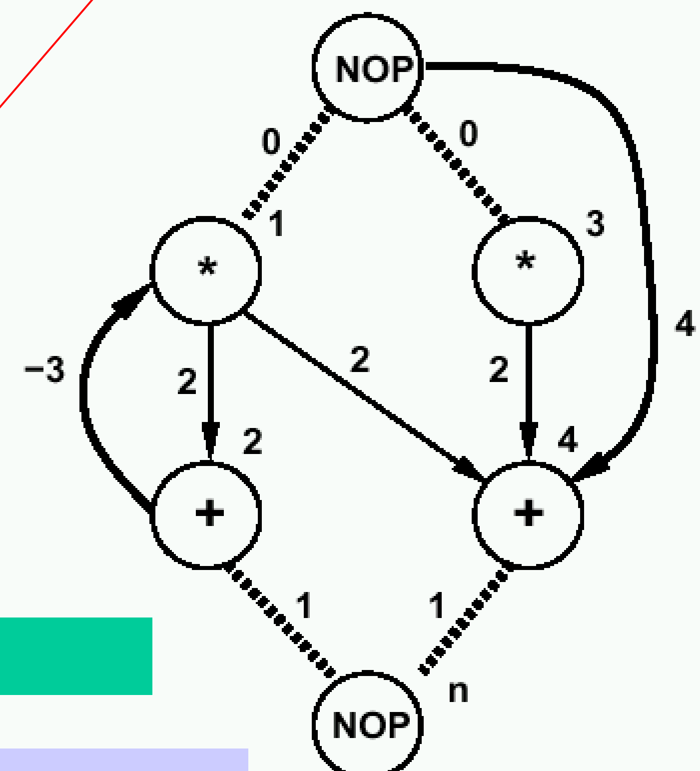
# Example of using constraint graph with minimum and maximum constraints

| Vertex | Start time |
|---|---|
| $v_0$ | 1 |
| $v_1$ | 1 |
| $v_2$ | 3 |
| $v_3$ | 1 |
| $v_4$ | 5 |
| $v_n$ | 6 |




Sequencing graph

explain




Constraint graph

# Methods for scheduling under detailed timing constraints

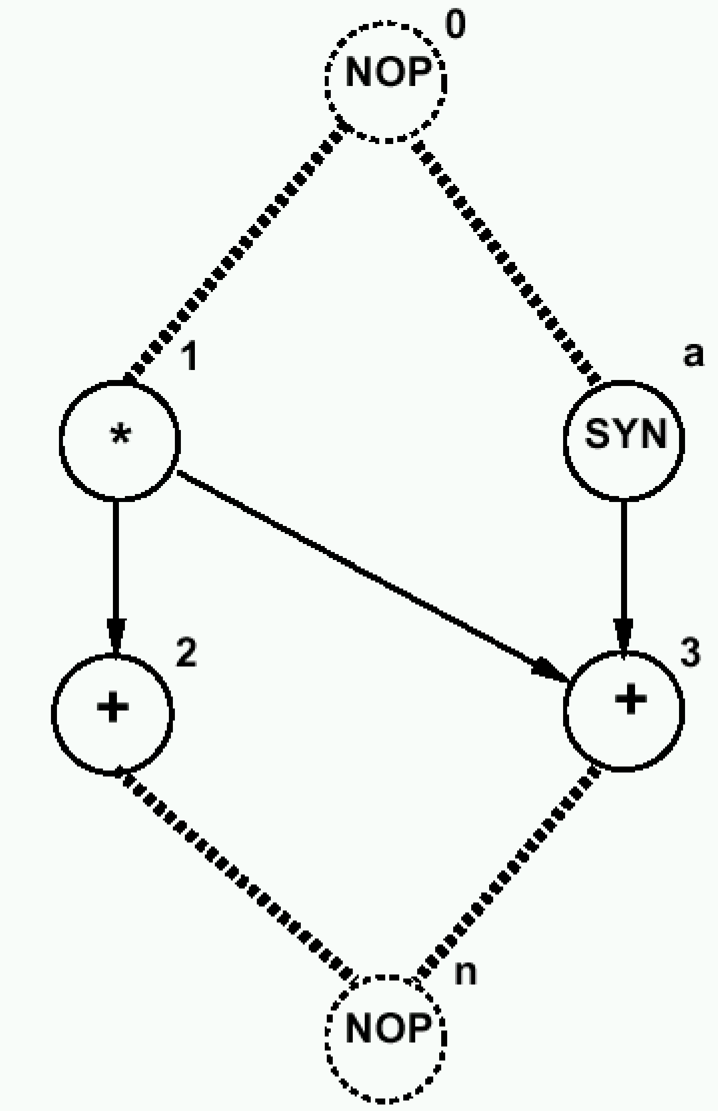
- Assumption:
  - All delays are fixed and known.
- Set of linear inequalities.
- **Longest path** problem.
- Algorithms for the longest path problem were discussed in Chapter 2:
  - Bellman-Ford,
  - Liao-Wong.

# Method for scheduling with unbounded-delay operations

- Unbounded delays:
  - Synchronization.
  - Unbounded-delay operations (e.g. *loops*).
- *Anchors.*
  - Unbounded-delay operations.
- Relative scheduling:
  - Schedule operations with **respect to the anchors**.
  - **Combine** schedules.

# Example of what?

- $t_3 = \max \{t_1 + d_1 ; t_a + d_a\}$



# Relative scheduling method

- For each vertex:
  - Determine ***relevant anchor set R(.).***
  - Anchors affecting start time.
  - Determine <u>time offset</u> *from anchors*.
- Start-time:
  - Expressed by:

$$t_i = \max_{a \in R(v_i)} \{ t_a + d_a + t_i^a \}$$

  - Computed only at run-time because delays of anchors are unknown.

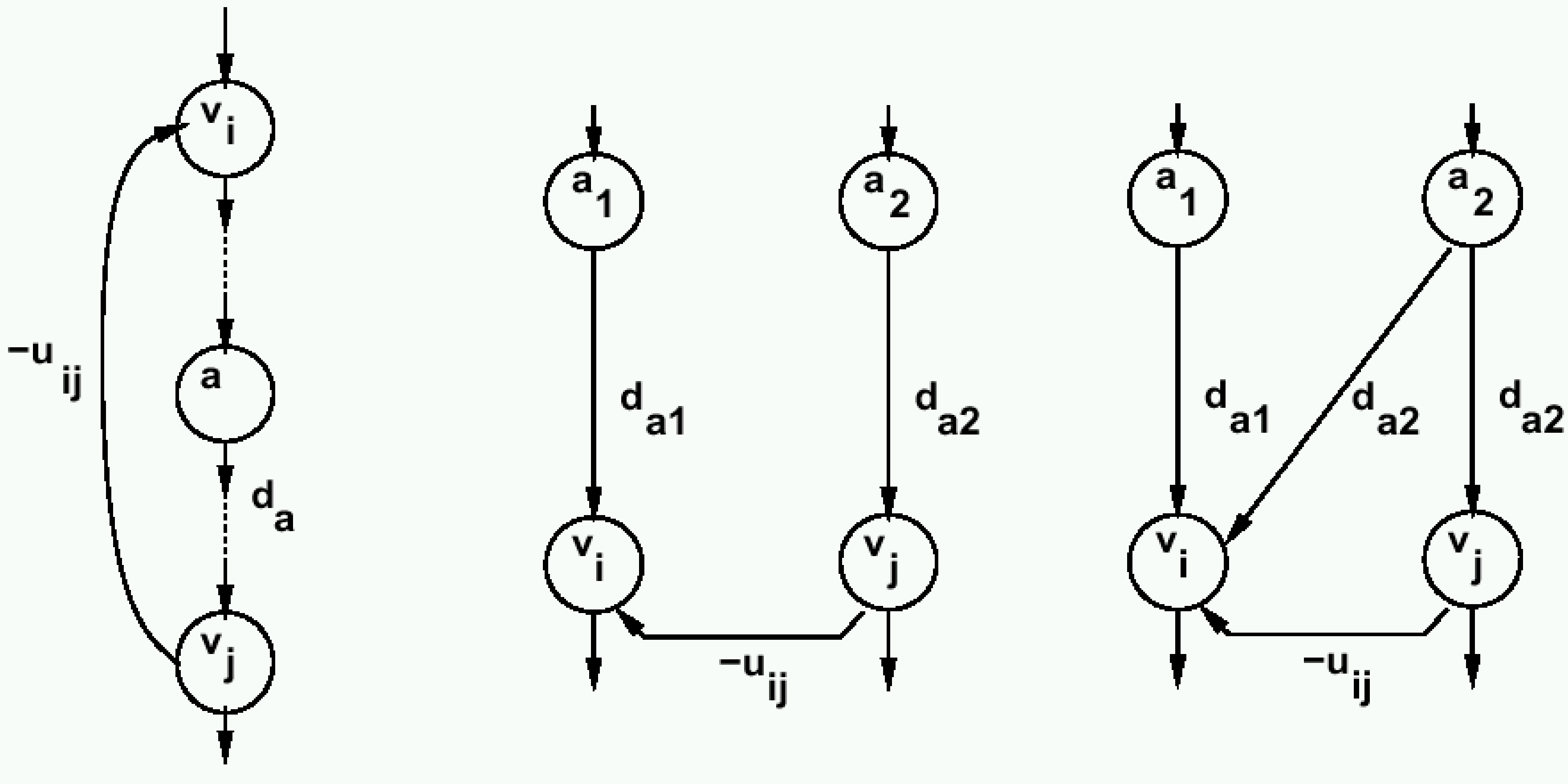
# Relative scheduling under timing constraints

- Problem definition:
  - Detailed timing constraints.
  - Unbounded delay operations.
- Solution:
  - May or may not exist.
  - Problem may be ill-specified.

# Relative scheduling under timing constraints

- Feasible problem:
  - A solution exists when unknown delays are zero.
- Well-posed problem:
  - A solution exists for any value of the unknown delays.
- Theorem:
  - A constraint graph can be made **well-posed** ***iff*** there are **no cycles** with unbounded weights.

# Example of Relative scheduling under timing constraints




(a) (b) (c)

- V
- V
- V

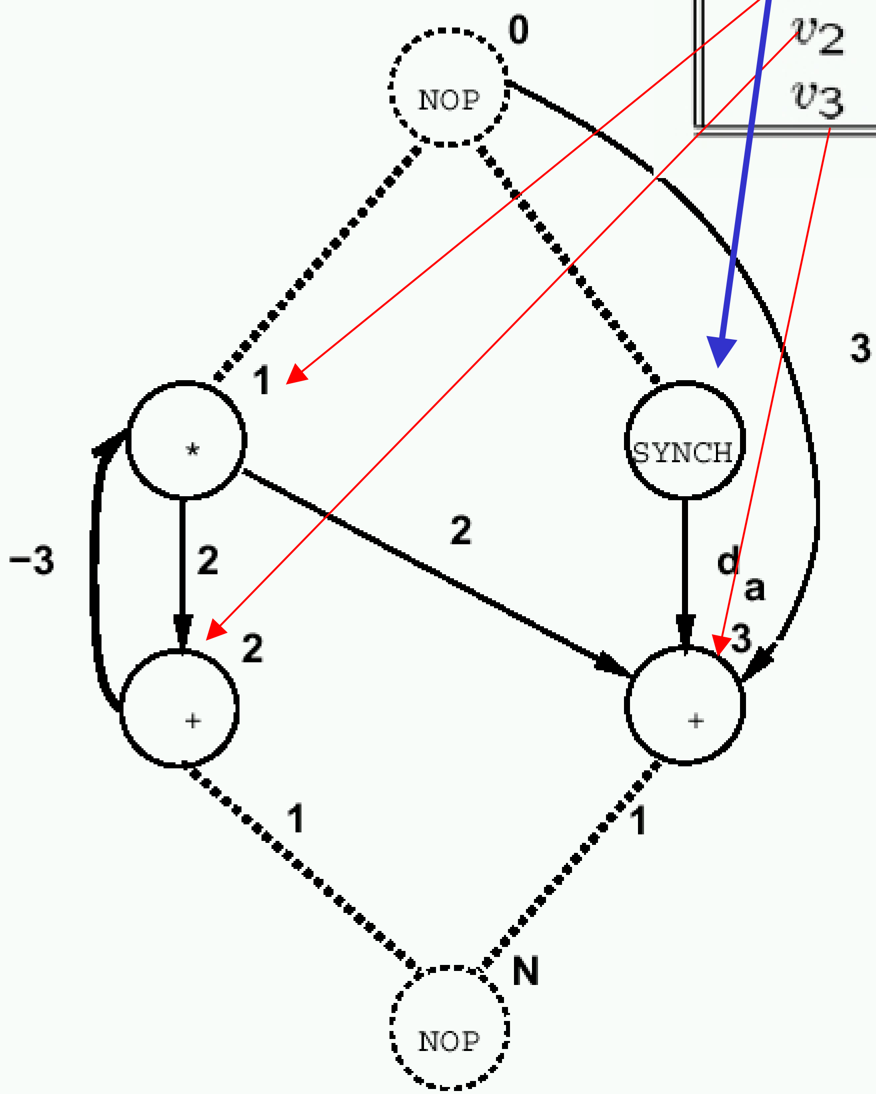
# Relative scheduling approach

- Analyze graph:
  - Detect anchors.
  - ***Well-posedness*** test.
  - Determine dependencies from anchors.
- Schedule ops with respect to relevant anchors:
  - Bellman-Ford, Liao-Wong, Ku algorithms.
- Combine schedules to determine start times:

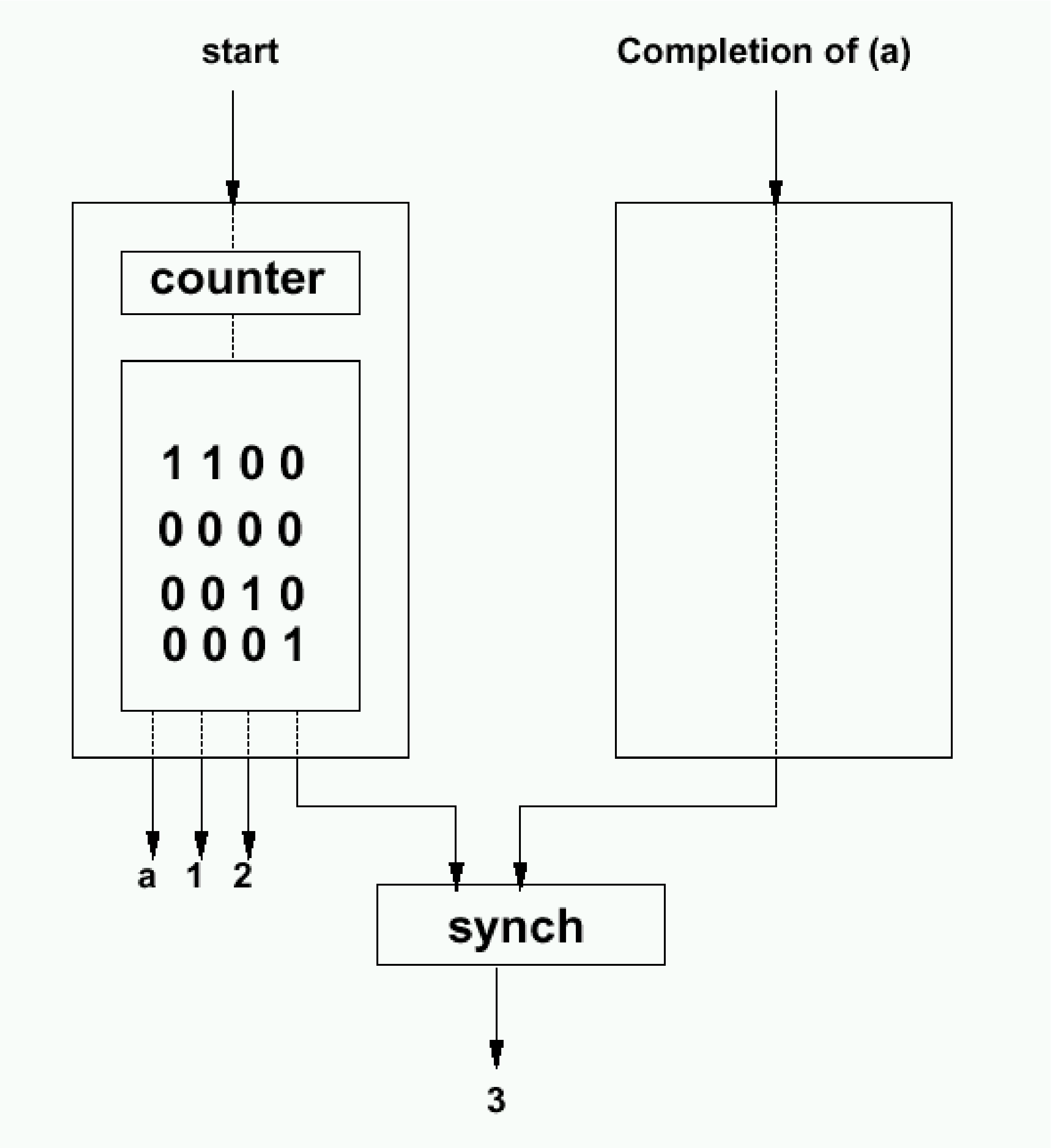
$$\boxed{t_i = \max_{a \in R(v_i)} \{ t_a + d_a + t_i^a \} \quad \forall i}$$

# Example of Relative scheduling

| Vertex $v_i$ | Relevant Anchor Set $R(v_i)$ | Offsets $t_0$ | $t_a$ |
|---|---|---|---|
| $a$ | $\{v_0\}$ | 0 | - |
| $v_1$ | $\{v_0\}$ | 0 | - |
| $v_2$ | $\{v_0\}$ | 2 | - |
| $v_3$ | $\{v_0, a\}$ | 3 | 0 |



# Example of control-unit synthesized for Relative scheduling



# Scheduling under resource constraints

- **Classical** scheduling problem.
  - Fix **area bound** - minimize **latency**.
- The amount of available resources affects the achievable latency.
- ***Dual*** problem:
  - Fix **latency bound** - minimize **resources.**
- Assumption:
  - All delays bounded and known.

# Minimum latency resource-constrained scheduling problem

- Given a set of operations **V** with integer delays **D** a partial order on the operations **E**, and upper bounds **{$a_k$ ; k = 1, 2,…,$n_{res}$}:**
- Find an integer labeling of the operations **φ : V --> $Z^+$** such that :
  - $t_i = \varphi(v_i)$,
  - $t_i \geq t_j + d_j \ \forall \ i, j \ s.t. \ (v_j, v_i) \in E$,
  - $|\{v_i | \mathcal{T}(v_i) = k$ and $t_i \leq l < t_i + d_i\}| \leq a_k$ $\forall$types $k = 1, 2, \ldots, n_{res}$ and $\forall$ steps $l$
  - and $t_n$ is *minimum*.

# Scheduling under resource constraints

- Intractable problem.
- Algorithms:
  - **Exact:**
    - **Integer linear** program.
    - **Hu** (restrictive assumptions).
  - **Approximate:**
    - **List** scheduling.
    - **Force-directed** scheduling.

# ILP formulation:

- Binary decision variables:

– $X = \{x_{il}; i = 1, 2, \ldots, n; l = 1, 2, \ldots, \overline{\lambda} + 1\}$.

– $x_{il}$, is TRUE only when operation $v_i$ starts in step $l$ of the schedule (i.e. $l = t_i$).

– $\overline{\lambda}$ is an upper bound on latency.

**Start time of operation $v_i$ :**

$$\sum_l l \cdot x_{il}$$

# ILP formulation constraints

- Operations start only once.

$$\sum_{l} x_{il} = 1 \quad i = 1, 2, \ldots, n$$

- Sequencing relations must be satisfied.

$$t_i \geq t_j + d_j \qquad \forall (v_j, v_i) \in E$$

$$\sum_{l} l \cdot x_{il} - \sum_{l} l \cdot x_{jl} - d_j \geq 0 \quad \forall (v_j, v_i) \in E$$

# ILP formulation constraints (cont)

***Resource bounds*** must be satisfied.

- Simple case (unit delay)

- $$\sum_{i:\mathcal{T}(v_i)=k} x_{il} \leq a_k \quad k = 1, 2, \ldots, n_{res}; \ \forall l$$

# ILP Formulation

Sequencing relations must be satisfied.

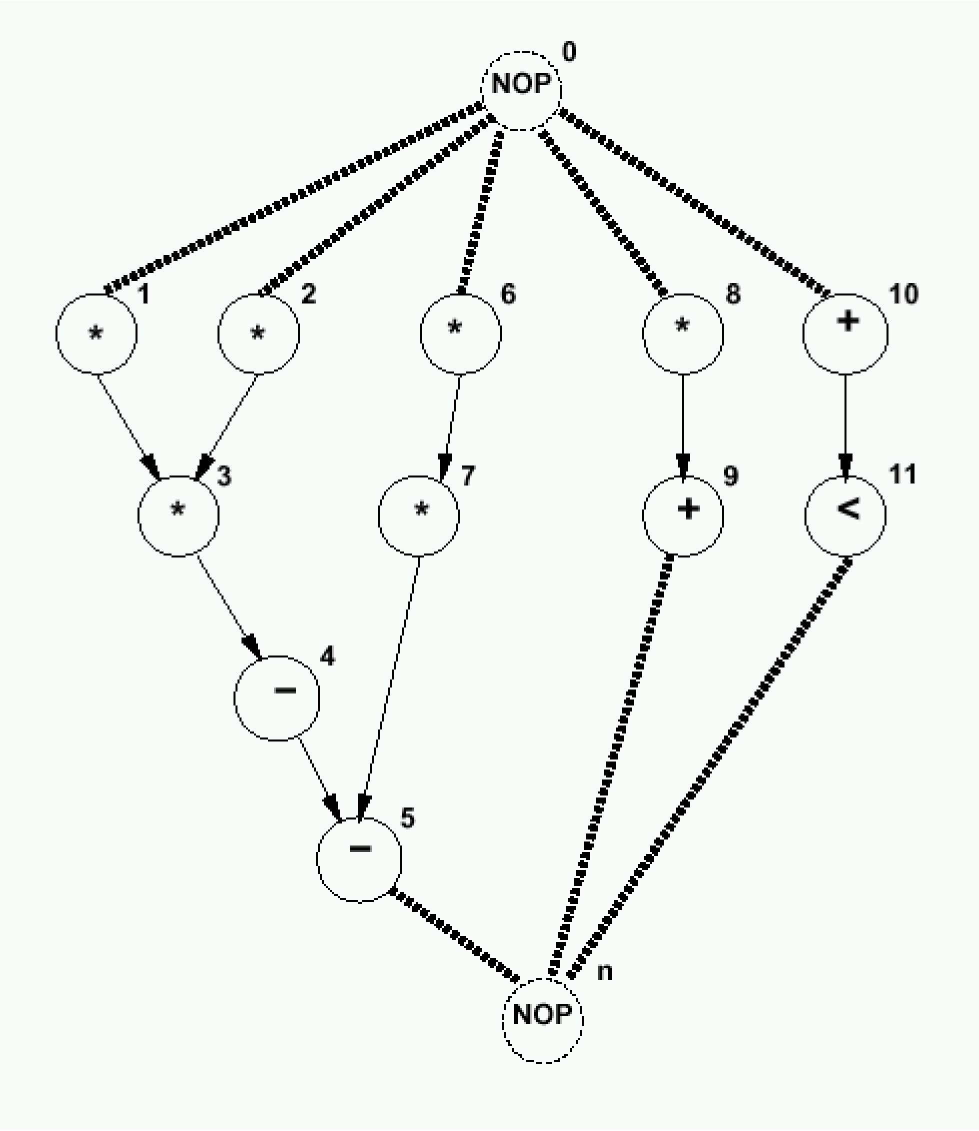
Operations start only once

$$
\begin{aligned}
\min \quad \|\mathbf{t}\| \quad & \textit{such that} \\
\sum_j x_{ij} &= 1 \quad i = 1, 2, \ldots, n \\
\sum_l l \cdot x_{il} - \sum_l l \cdot x_{jl} - d_j &\geq 0 \quad i, j = 1, 2, \ldots, n, (v_j, v_i) \in E \\
\sum_{i:\mathcal{T}(v_i)=k} \; \sum_{m=l-d_i+1}^{l} x_{im} &\leq a_k \quad k = 1, 2, \ldots, n_{res}; l = 0, 1, \ldots, t_n
\end{aligned}
$$

*Resource bounds* must be satisfied

# Example of ILP Formulation

- Resource constraints:
  - 2 ALUs, 2 Multipliers.
  - $a_1 = 2$, $a_2 = 2$.
- Single-cycle operation.
  - $d_i = 1 \ \forall i$.

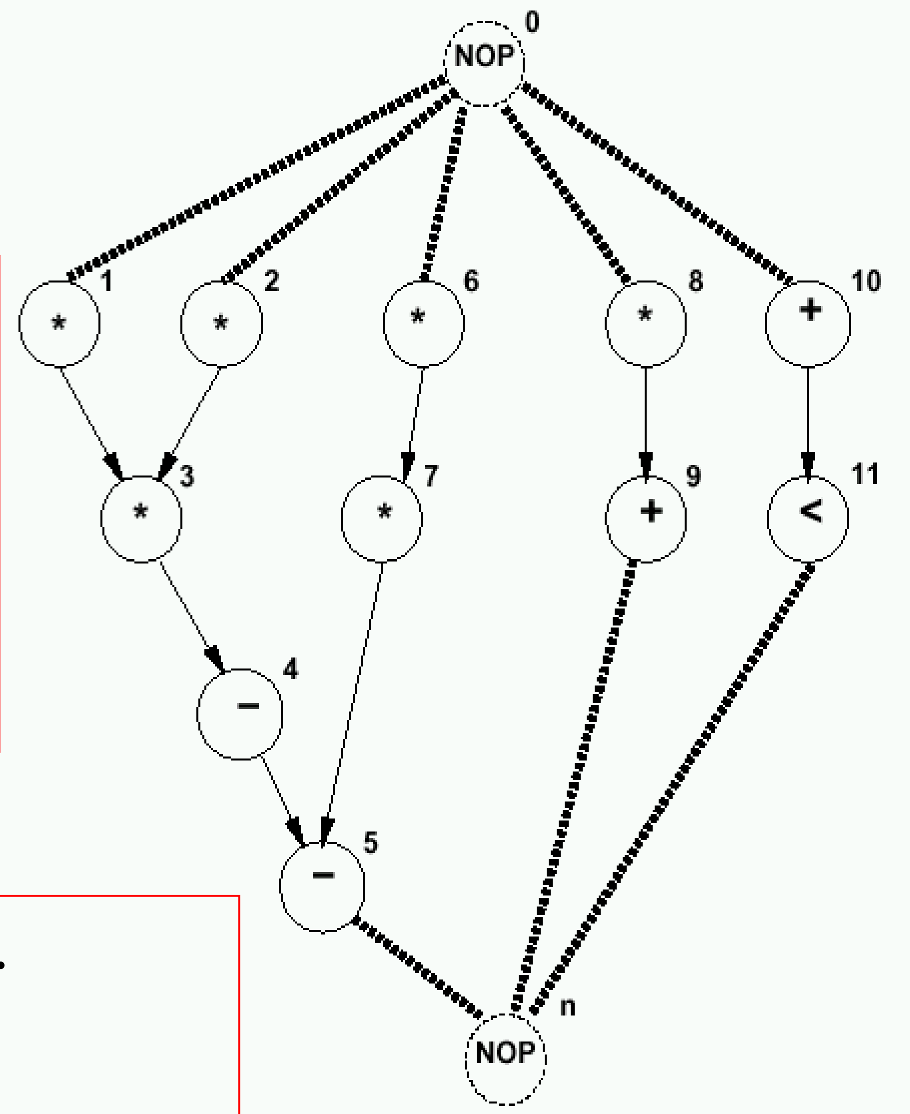


# Example ILP

Operations start only once.

- $x_{11} = 1$
- $x_{61} + x_{62} = 1$
- ...

- **Sequencing relations must be satisfied.**
  - $x_{61} + 2\,x_{62} - 2\,x_{72} - 3x_{73} + 1 \leq 0$
  - $2\,x_{92} + 3\,x_{93} + 4\,x_{94} - 5\,x_{N5} + 1 \leq 0$
  - .....

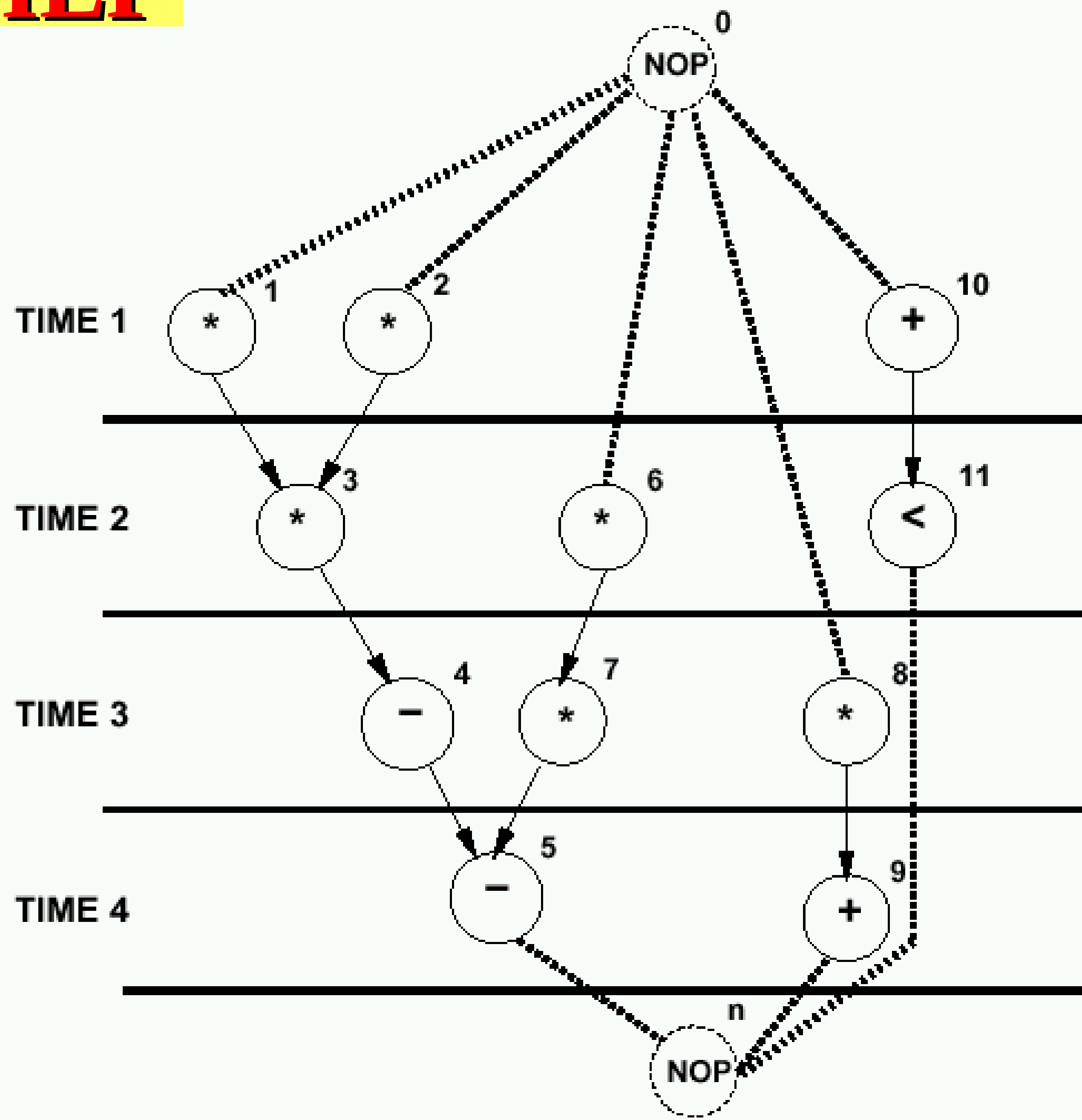



Resource bounds must be satisfied.

- $x_{11} + x_{21} + x_{61} + x_{81} \leq 2$
- $x_{32} + x_{62} + x_{72} + x_{82} \leq 2$
- ...

# Example ILP

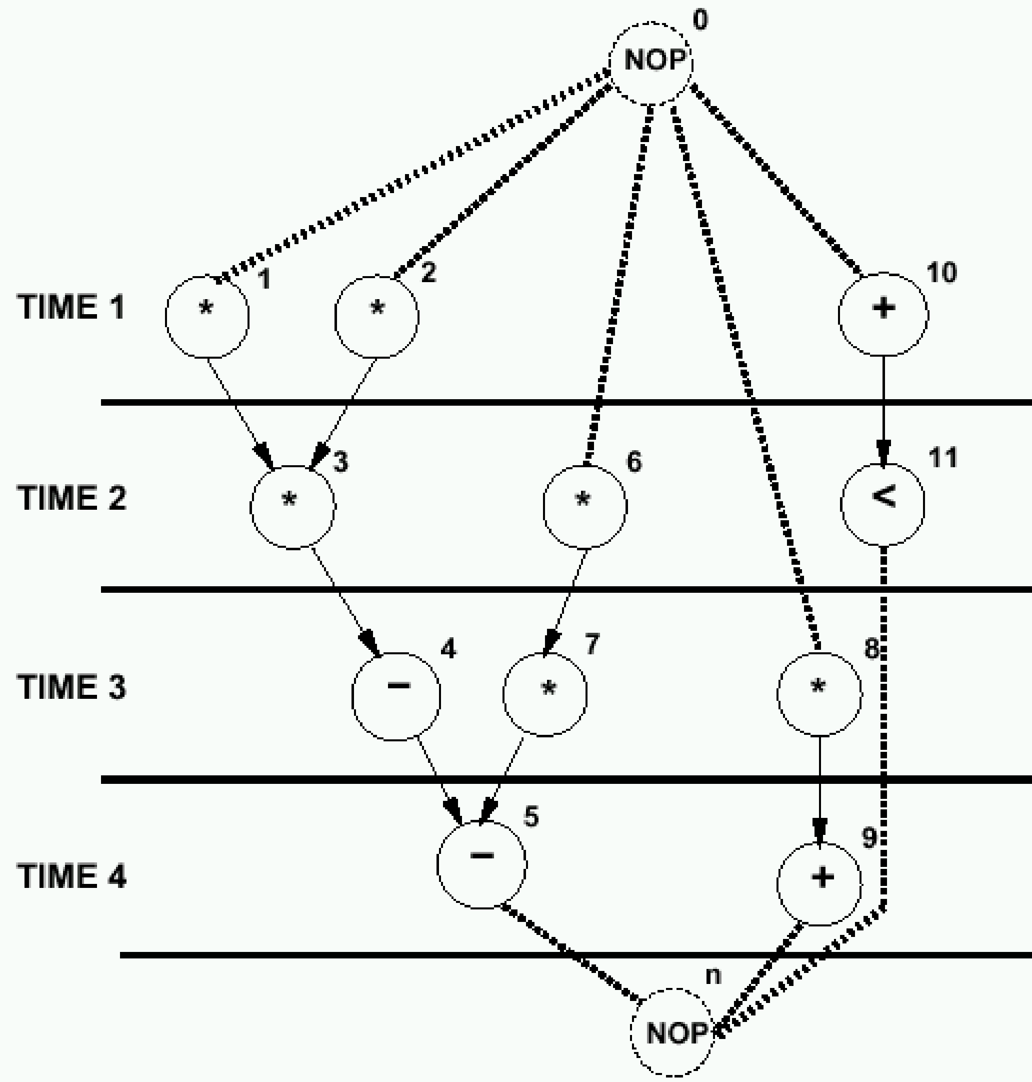
- Solution
- latency 4
- multipliers =2
- ALU =2



# Dual ILP formulation

- Minimize resource usage under latency constraints.
- Additional constraint:
  - Latency bound must be satisfied.
  - $$\boxed{\sum_{l} l \; x_{nl} \leq \overline{\lambda} + 1}$$
- Resource usage is unknown in the constraints.
- Resource usage is the objective to minimize.

# Example




**Multipliers = 2**

**ALUs = 2**

**2 * 5 + 2*1= 12= cost of solution**

- Multiplier area = 5. ALU area = 1.
- Objective function: $5a_1 + a_2$ .

# ILP Solution

- Use **standard ILP** packages.
- Transform into LP (linear programming) problem **[Gebotys].**
- ***Advantages:***
  - Exact method.
  - Others constraints can be incorporated.
- ***Disadvantages:***
  - Works well up to few thousand variables.