Boolean Algebra

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Outline

- Set, Relations, and Functions
- Partial Orders
- Boolean Functions
- Don’t Care Conditions
- Incomplete Specifications

Set Notation

$v \in V$ Element v is a member of set V

$v \notin V$ Element v is not a member of set V

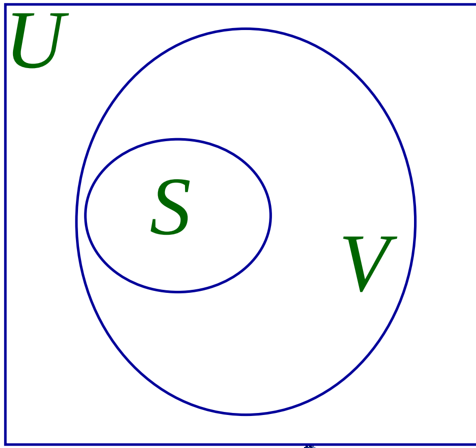
$|V|$ Cardinality (number of members) of set V

$S \subseteq V$ Set S is a subset of V

$\varnothing$ The empty set (a member of all sets)

S' The complement of set S

U The universe: $S' = U - S$



Set Notation

- **Inclusion ($\subset$)**
- **Proper Inclusion ($\subseteq$)**
- **Complementation**
- **Intersection ($\cap$)**
- **Union ($\cup$)**
- **Difference**

Power Sets

$$2^V = \{S | S \subseteq V\}$$

2^V The power set of set V (the set of all subsets of set V

$|2^V| = 2^{|V|}$ The cardinality of a power set is a power of 2

Power Sets

$V = \{0,1,2\}$ 3-member set

$2^V = \{\varnothing,$ 1 subset with 0 members

$\{0\},\{1\},\{2\},$ 3 subsets with 1 members

$\{0,1\},\{0,2\},\{1,2\},$ 3 subsets with 2 members

$\{0,1,2\}\}$ 1 subset with 3 members

$$|2^V| = 2^{|V|} = 2^3 = 8$$

Power sets are Boolean Algebras

Cartesian Products

The **Cartesian Product** of sets A and B is denoted A x B

Suppose $A = \{0,1,2\}, \quad B = \{a,b\}$, then

$$A \times B = \{(0,a),(0,b),(1,a),(1,b),(2,a),(2,b)\}$$

$A = \{0,1,2\}$ Set A is unordered

$(1,b)$ ( ) denotes Ordered Set

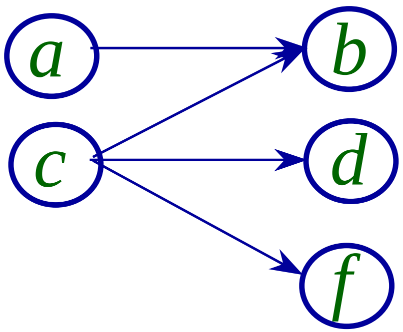
Binary Relations

- The **Cartesian Product** of sets A and B is denoted A x B
- A x B consists of all possible ordered pairs (a,b) such that $a \in A$ and $b \in B$
- A subset $R \subseteq A \times B$ is called a **Binary Relation**
- Graphs, Matrices, and Boolean Algebras can be viewed as binary relations

Binary Relations as Graphs or Matrices

$$E = \{ab, cb, cd, cf\} \subseteq A \times B$$

$$A = \{a, c\} \quad B = \{b, d, f\}$$



	b	d	f
a	1	0	0
c	1	1	1

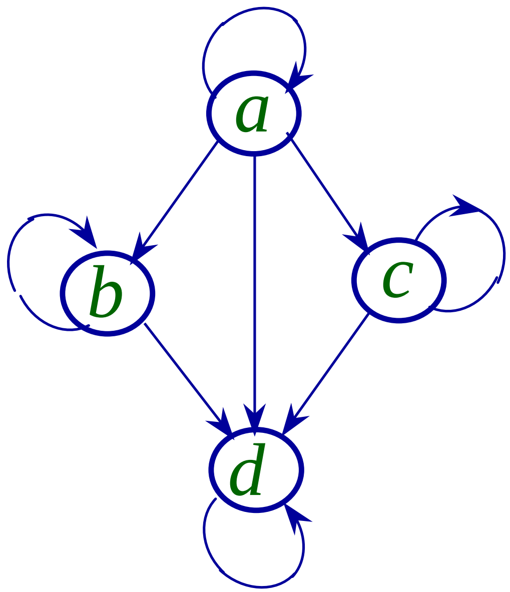
Bipartite Graph

Rectangular Matrix

Binary Relations as Graphs or Matrices

$$E = \{ab, ac, ad, bd, cd\} \cup \{aa, bb, cc, dd\} \subseteq V \times V$$

$$V = \{a, b, c, d\}$$



Directed Graph

a	**1**	**1**	**1**	**1**
b	**0**	**1**	**0**	**1**
c	**0**	**0**	**1**	**1**
d	**0**	**0**	**0**	**1**

Square Matrix

Properties of Binary Relations

- A binary relation $R \subseteq V \times V$ can be
  - reflexive, and/or
  - transitive, and/or
  - symmetric, and/or
  - antisymmetric
- We illustrate these properties on the next few slides

Notation for Binary Relations

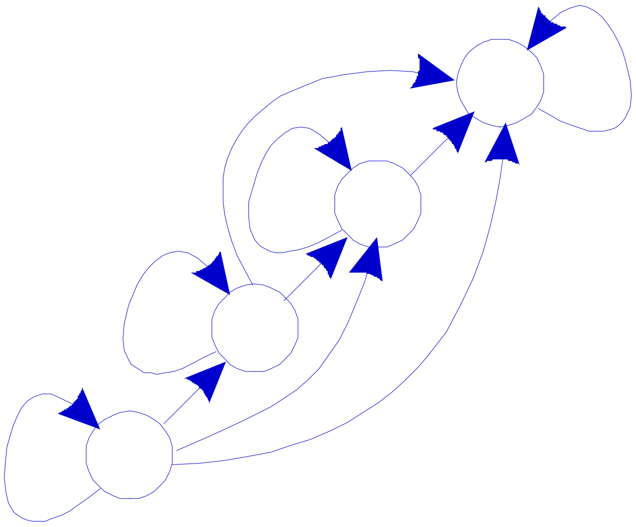
If $R \subseteq A \times B$, we say that A is the **domain** of the relation R, and that B is the **range**.

If $(a,b) \in R$, we say that the pair is in the relation R, or aRb.

Missing Figure 3.2, Page 80 3.1.3

$$\leq \;\subseteq\; A^2 = \{1,2,3,4\}^2$$

	1	2	3	4
1	≤	≤	≤	≤
2		≤	≤	≤
3			≤	≤
4				≤



Graph View

Example: “Less than or Equal”

$$A = B = \{0,1,2,\ldots\},$$
$$R \subseteq A \times B = \text{"}\leq\text{"}$$

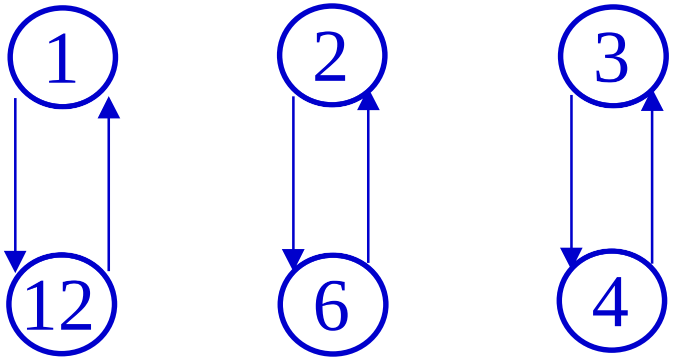
$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} \leq & \leq & \leq & \leq & \cdots \\ 0 & \leq & \leq & \leq & \\ 0 & 0 & \leq & \leq & \\ 0 & 0 & 0 & \leq & \\ \vdots & & & & \ddots \end{bmatrix} \end{array}$$

Example: "*a* times *b* = 12"

$$V = \{0,1,2,\ldots\},$$

$$R \subseteq V \times V = \{(u,v) | u \times v = 12\}$$

$$R = \{(1,12),(2,6),(3,4),(4,3),(6,2),(12,1)\}$$

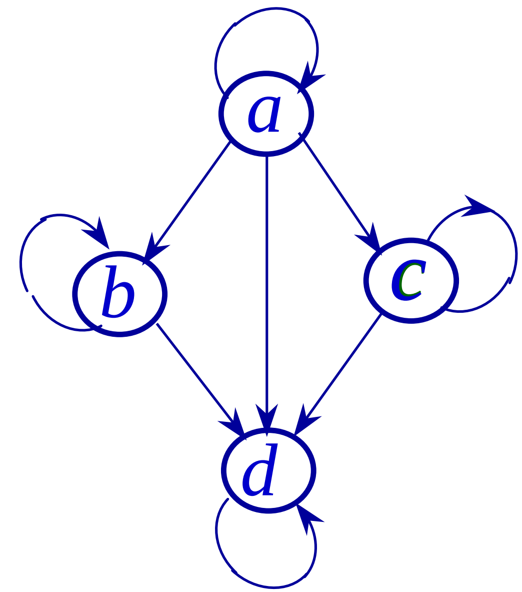


Reflexive Binary Relations

$$V = \{a, b, c, d\}$$

$$R \subseteq V \times V$$

$$v \in V \Rightarrow vRv$$



A binary relation $R \subseteq V \times V$ is reflexive if and only if $(v, v) \in R$ for every vertex $v \in V$

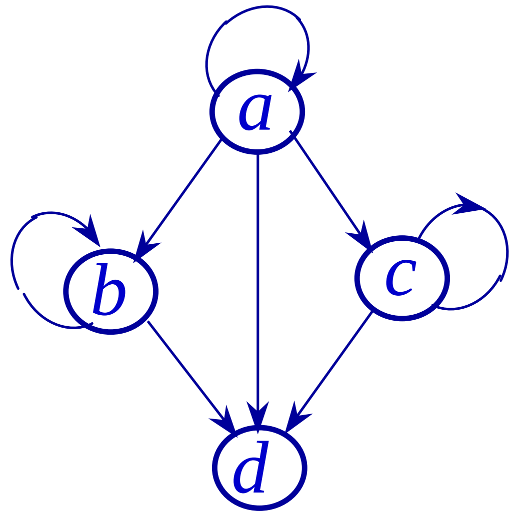
$$\begin{matrix} a \\ b \\ c \\ d \end{matrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Non-Reflexive Binary Relations

$$V = \{a, b, c, d\}$$

$$R \subseteq V \times V$$

$$\exists v \in V \ni \neg vRv$$

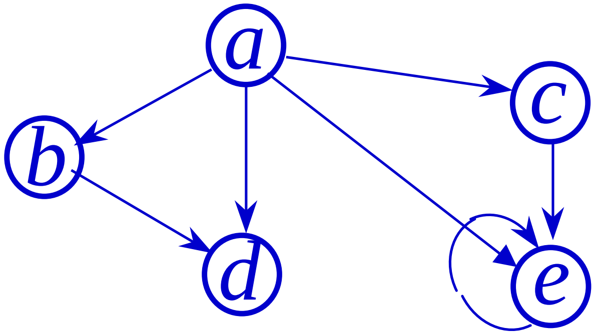


Non-Reflexivity implies that there exists $v \in V$ such that $(v,v) \notin R$. Here d is such a v.

$$\begin{array}{c} a \\ b \\ c \\ d \end{array} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Transitive Binary Relations

If $u, v, w \in V$, and uRv, and vRw, then uRw.



A binary relation is transitive if and only if every *(u,v,w)* path is triangulated by a direct *(u,w)* edge.

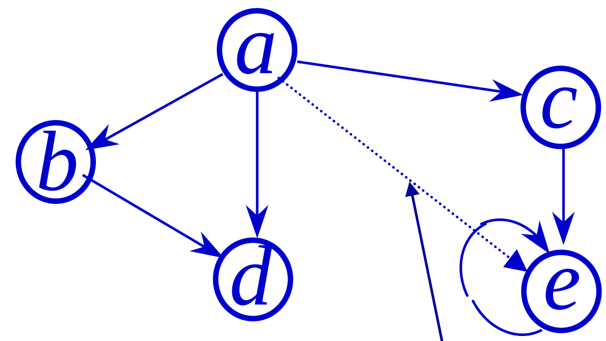
This is the case here, so *R* is transitive.

Non-Transitive Binary Relations

$$R \subseteq V \times V$$

$\exists u, v, w \in V$, such that

uRv, vRw, but $\neg uRw$



A binary relation is not transitive if there exists a path from *u* to *w*,, through *v* that is not triangulated by a direct *(u,w)* edge

Here *(a,c,e)* is such a path.

Edge *(a,e)* is missing

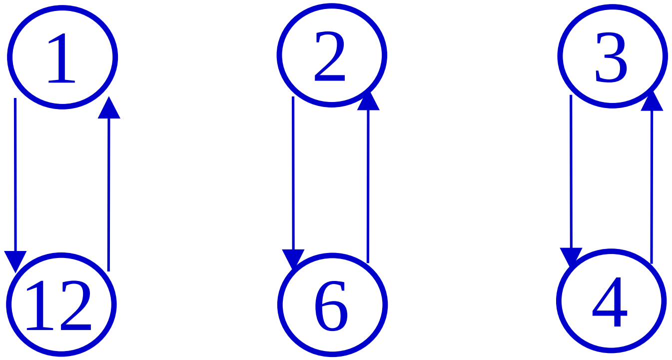
Symmetric Binary Relations

$$R \subseteq V \times V$$

$$(u,v) \in R \Rightarrow (v,u) \in R$$

$$(uRv \Rightarrow vRu)$$

A binary relation is symmetric if and only if every (u,v) edge is reciprocated by a (v,u) edge



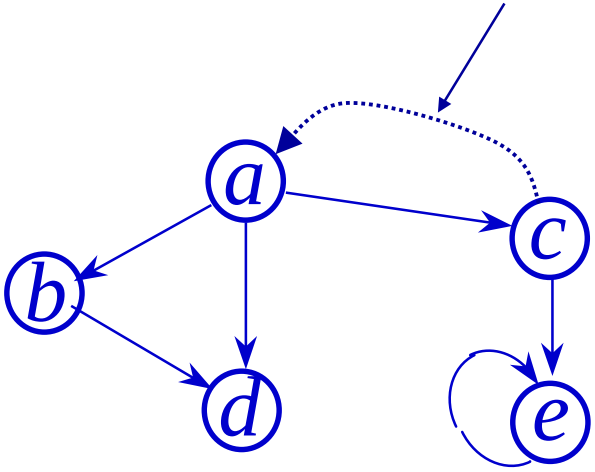
Non-Symmetric Binary Relations

$$R \subseteq V \times V$$

$$\exists (u,v) \in R \text{ such that } (v,u) \notin R$$

A binary relation is non-symmetric if there exists an edge (u,v) not reciprocated by an edge (v,u)

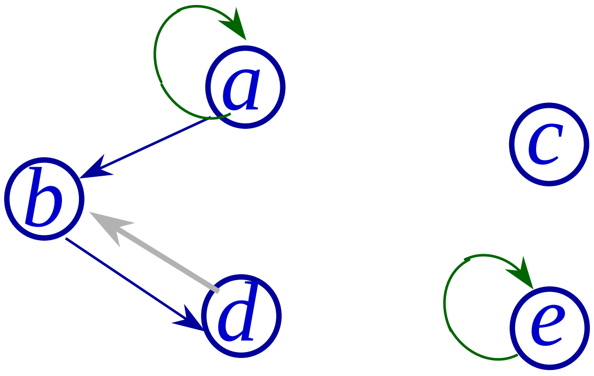
Edge (c,a) is missing



Antisymmetric Binary Relations

$$\forall (u,v) \in R, (uRv, vRu) \Rightarrow (v = u)$$

A binary relation is anti-symmetric if and only if no (u,v) edge is reciprocated by a (v,u) edge $v = u$



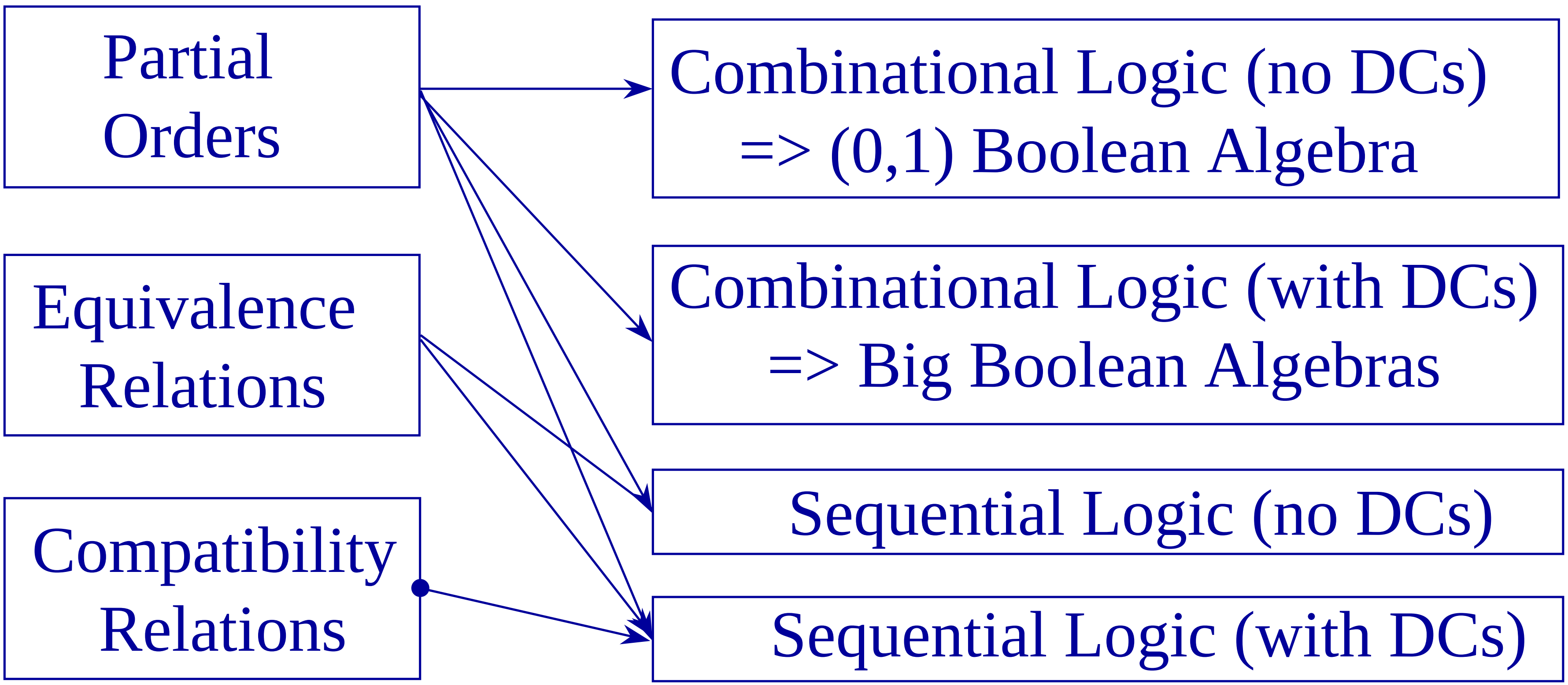
Not antisymmetric if any such edge is reciprocated : here $b \rightarrow d, \ d \rightarrow b$

Functions

- A function *f* are binary relations from set *A* (called the domain) to *B* (called the range)
- But, it is required that each *a* in *A* be associated with exactly 1 *b* in *B*
- For functions, it cannot be true that both *(a,b)* in *R* and *(a,c)* in *R*, *b* different from *c*

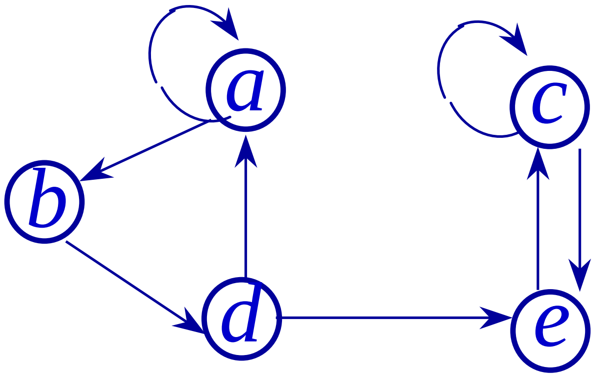
The Binary Relation of Relations to Synthesis/Verification

*DC=don’t care



The Path Relation $\xrightarrow{*}$

$G=(V,E)$

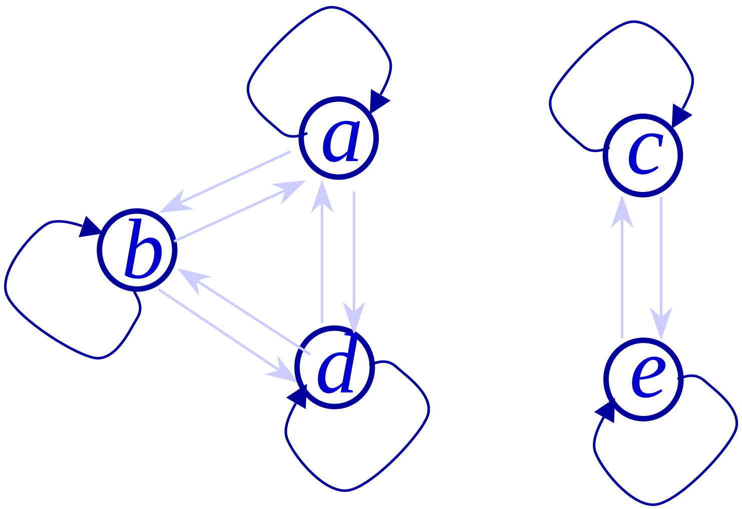


This graph defines path relation

$a \xrightarrow{*} \{a,b,c,d,e\}$
$b \xrightarrow{*} \{c,d,e\}$
$c \xrightarrow{*} \{c,e\}$
$d \xrightarrow{*} \{c,e\}$
$e \xrightarrow{*} \{c,e\}$

Relation $\xrightarrow{*}$ is sometimes called "Reachability"

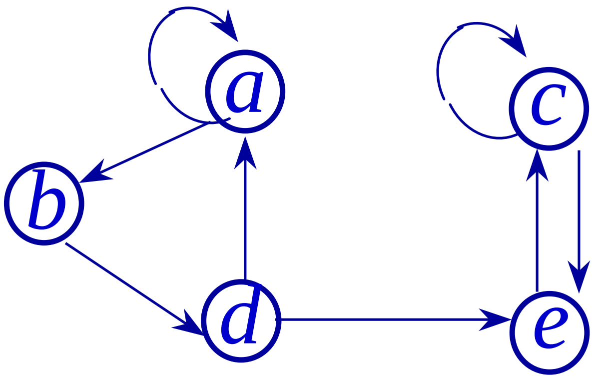
Equivalence Relations



Note *R* is reflexive, symmetric, and transitive

Equivalence Relations

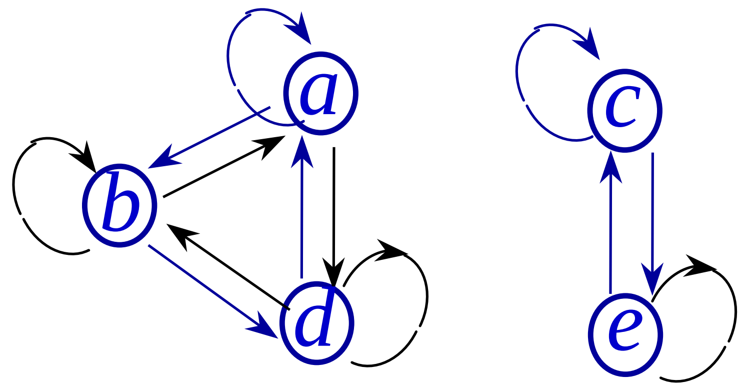
$G=(V,E)$



This graph defines path relation $\xrightarrow{*}$

$R=\{(u,v) \mid v \xrightarrow{*} u,\ u \xrightarrow{*} v\}$

$G=(V,R)$



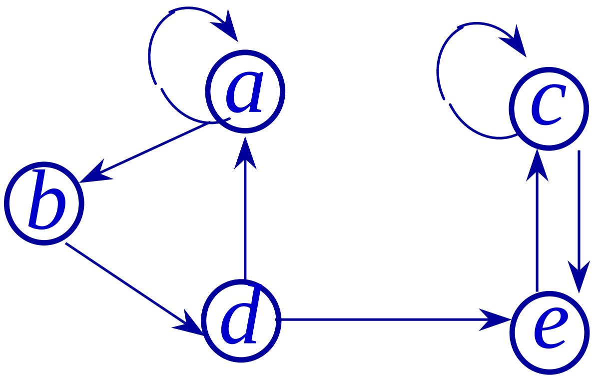
This graph defines R

NOT an equivalence relation: E

An equivalence relation: R

The Cycle Relation

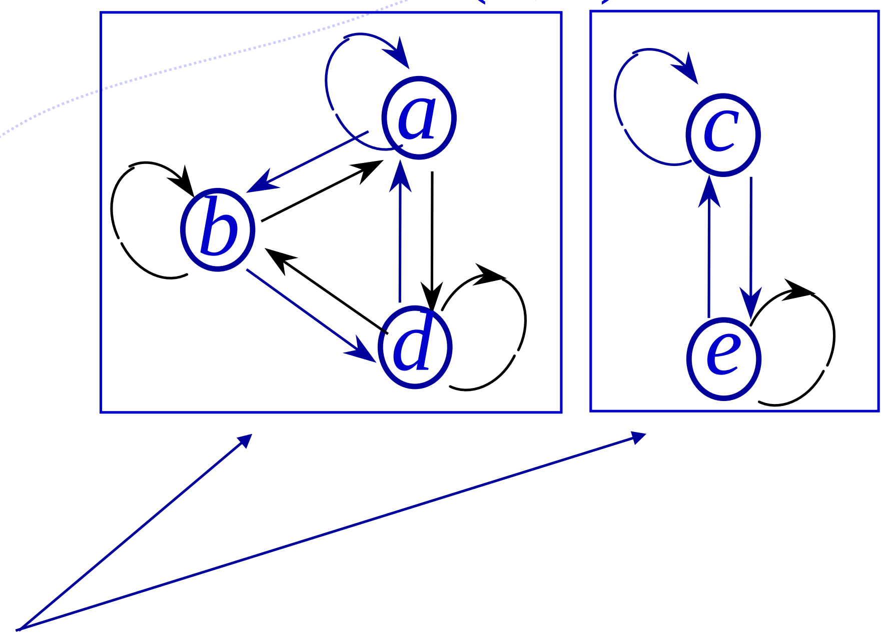
$G=(V,E)$



This graph defines path relation $\overset{*}{\longleftrightarrow}$

$C=\{(u,v)|v \overset{*}{\longleftrightarrow} u\}$

$G=(V,R)$



These are called the "Strongly Connected Components" of G

Refinement

- Given any set B a **partition** of B is a set of subsets $B_i \subseteq B$ with two properties
  - $B_i \cap B_j = \phi$ for all $i \neq j$
  - $\cup_i B_i = B$

- Given two partitions P^1 and P^2 of a set S, P^1 is a **refinement** of P^2 if each block $B^1{}_i$ of P^1 is a subset of some block of π_2

Other Binary Relations

- **Partial Orders (Includes Lattices, Boolean Algebras)**
  - **Reflexive**
  - **Transitive**
  - **Antisymmetric**

- **Compatibility Relations**
  - **Reflexive**
  - **Not Transitive--Almost an equivalence relation**
  - **Symmetric**

Functions

- A **function** f from A to B written $f : A \rightarrow B$ is a rule that associates exactly one element of B to each element of A
  - A relation from A to B is a function if it is right-unique and if every element of A appears in one pair of the relation
  - A is called the domain of the function
  - B is called the co-domain (range)
- If $y=f(x) : A \rightarrow B$, y is **image** of x
  - Given a domain subset $C \subseteq A$
    $IMG(f,C) = \{ y \in B \mid \exists x \in C \ni y = f(x) \}$
  - **preimage** of C under f
    $PRE(f,C) = \{ x \in A \mid \exists y \in C \ni y = f(x) \}$
- A function f is one-to-one (injective) if $x \neq y$ implies $f(x) \neq f(y)$
- A function f is onto (surjective) if for every $y \in B$, there exists an element $x \in A$, such that $f(x) = y$

Algebraic Systems

- **The pair $(V, \leq)$ is called an algebraic system**
- **V is a set, called the carrier of the system**
- **$\leq$ is a relation on $V \times V = V^2$**
  ($\subseteq$, $\Rightarrow$ are similar to $\leq$)
- **This algebraic system is called a partially ordered set, or poset for short**

Partially Ordered Sets

- A **poset** has two operations, $\bullet$ and $+$, called **meet** and **join** (like **AND** and **OR**)
- Sometimes written $(V, \leq, \cdot, +)$ or $(V^2, \leq, \cdot, +)$ even $(V, \cdot, +)$, since $\leq$ is implied

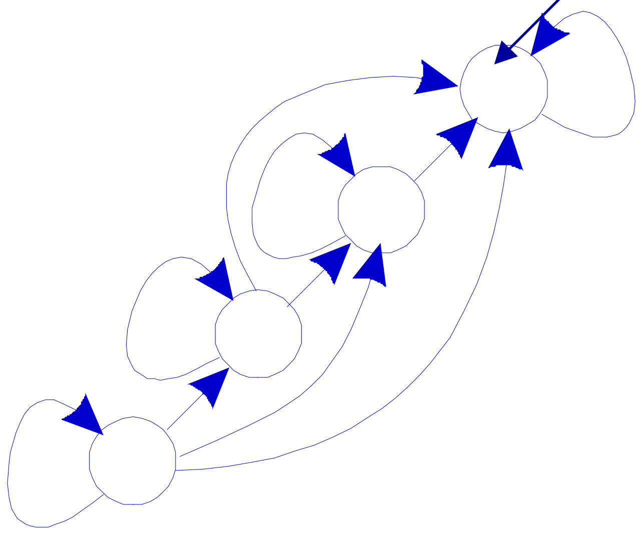
Integers (Totally Ordered):

Missing Figure 3.2

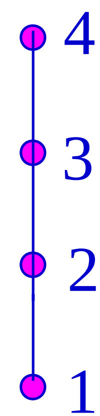
$$\leq \;\subseteq\; A^2 = \{1,2,3,4\}^2$$

	1	2	3	4
1	≤	≤	≤	≤
2		≤	≤	≤
3			≤	≤
4				≤

Matrix View



Graph View



Hasse Diagram

Hasse Diagram obtained deleting arrowheads and redundant edges

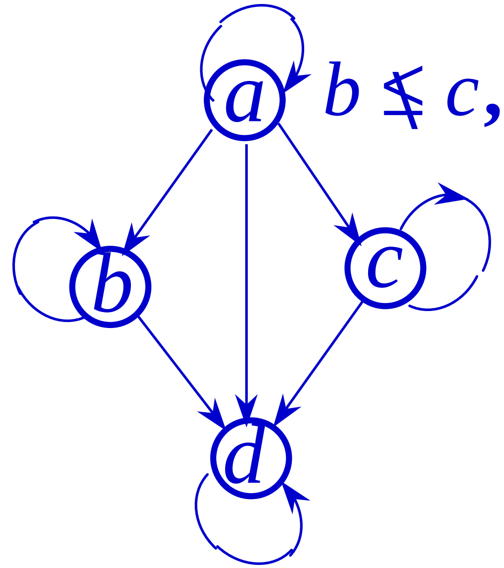
Posets and Hasse Diagrams

$(V, \leq, \cdot, +)$

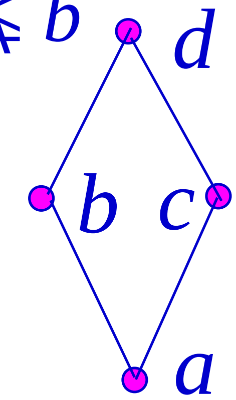
$V = \{a, b, c, d\}$

$\leq = \{(a,a), (a,b), (a,c), (a,d),$
$(b,b), (b,d),$
$(c,c), (c,d), (d,d)\}$

$b \nleq c, \; c \nleq b$



Relation $\leq$



Hasse Diagram

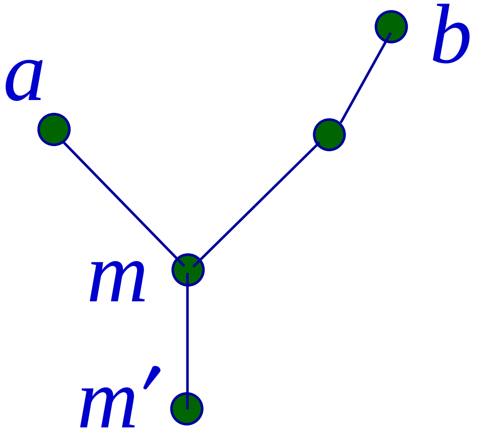
$\leq$:distance from top

$\leq$ is refelxive, antisymmetric, and transitive: a **partial order**

Meet

An element m of a poset P is a **lower bound** of elements a and b of P, if $m \le a$ and $m \le b$.

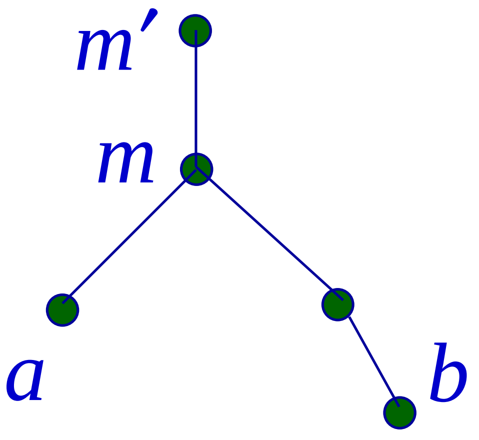
m is the **greatest lower bound** or **meet** of elements a and b if m is a lower bound of a and b and, for any m' such that $m' \le a$ and also $m' \le b$, $m' \le m$.



Join: the Dual of Meet

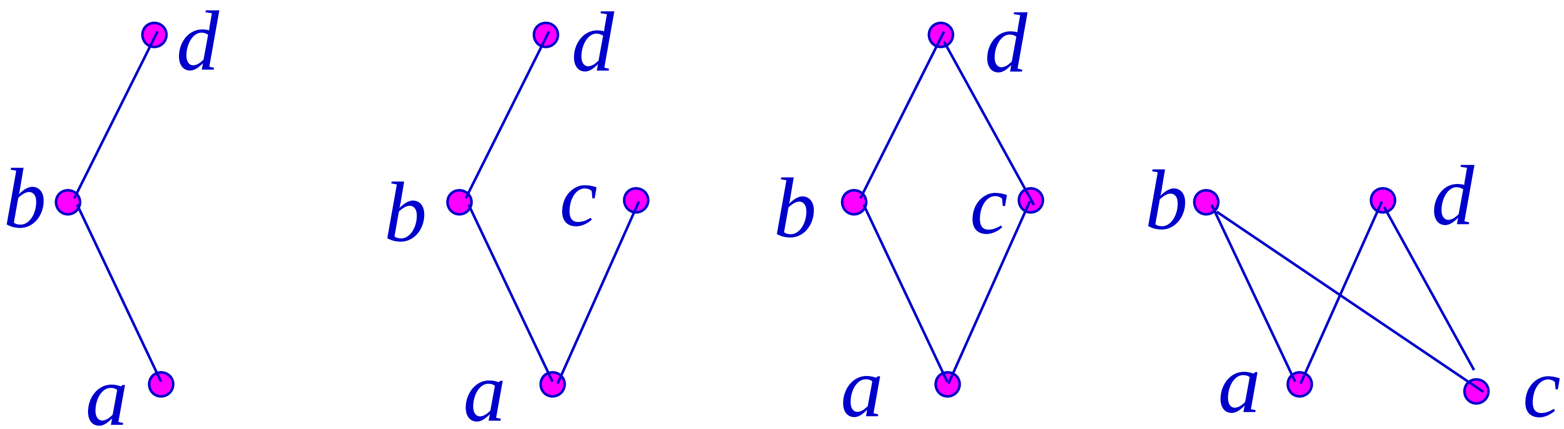
An element m of a poset P is a **upper bound** of elements a and b of P, if $a \leq m$ and $b \leq m$.

m is the **least upper bound** or **join** of elements a and b if m is an upper bound of a and b and, for any m' such that $a \leq m'$ and also $b \leq m'$, $m \leq m'$.



Meet (·) and Join (+)

posets:



$a \cdot b = a$ $\quad c \cdot b = a$ $\quad c \cdot b = a$ $\quad c \cdot a = ?$

$a + b = b$ $\quad c + b = ?$ $\quad c + b = d$ $\quad d + b = ?$

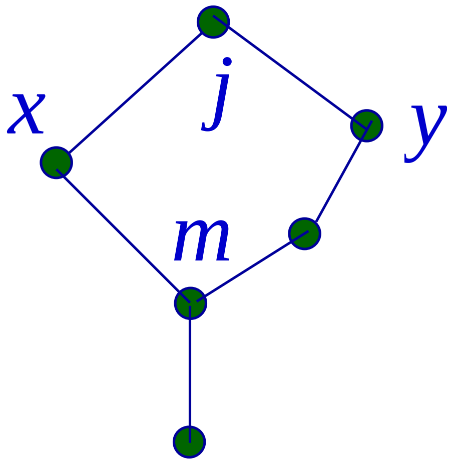
POSET Properties

Theorem 3.2.1 If x and y have a greatest lower bound (meet), then

$$x \geq x \cdot y$$

Similarly, if x and y have a least upper bound (join), then

$$x \leq x + y$$



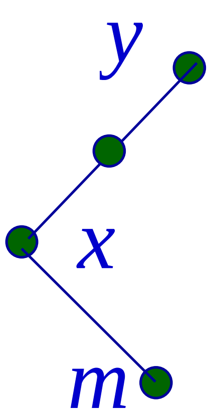
Proof: Since meet exists, $x \geq x \cdot y$ by definition. Also, since join exists, $x \leq x + y$ by definition.

More POSET Properties

Theorem 3.2.2 $x \leq y \Leftrightarrow x \cdot y = x$

Proof ($\Rightarrow$): This means assume $x \leq y$.

- x is a lower bound of x and y (by Def.)
- x is also the meet of x and y



Proof: by contradiction. Suppose $x \neq x \cdot y$. Then $\exists m \neq x$ **such that** $x \leq m$ where $m = x \cdot y$. But since m was a lower bound of x and y, $m \leq x$ as well. Thus $m = x$, by the anti-symmetry of posets.

Proof ($\Leftarrow$): From Definition of meet

Well-Ordered

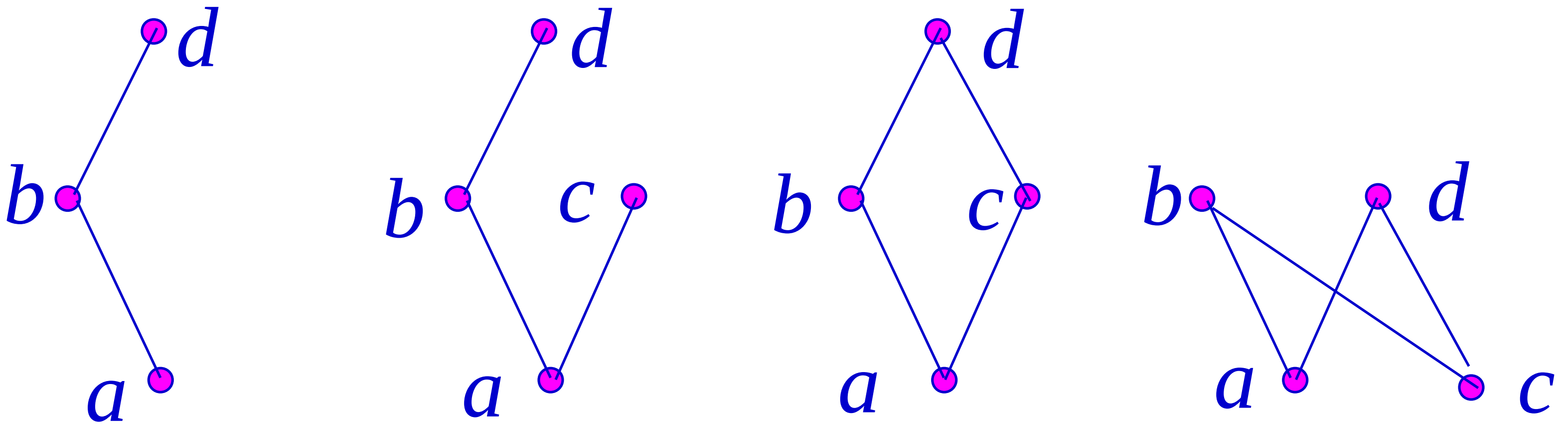
- **If all pairs of elements of a poset are comparable, then the set is totally ordered**
- **If every non-empty subset of a totally ordered set has a smallest element, then the set is well-ordered**
  - **e.g.) Natural numbers**

- **Mathematical Induction**
  - **Given, for all n $\in$ N, propositions P(n), if**
    - **P(0) is true**
    - **for all n>0, if P(n-1) is true then P(n) is true**
  - **then, for all n $\in$ N, P(n) is true**

Lattices

- **Lattice: a poset with both meet and join for every pair of elements of the carrier set**
- **Boolean Algebra: a distributed and complemented lattice**
- **Every lattice has a unique minimum element and a unique maximum element**

Lattices and Not Lattices



$a \cdot b = a$

$a + b = b$

(lattice)

$c \cdot b = a$

$c + b = ?$

$c \cdot b = a$

$c + b = d$

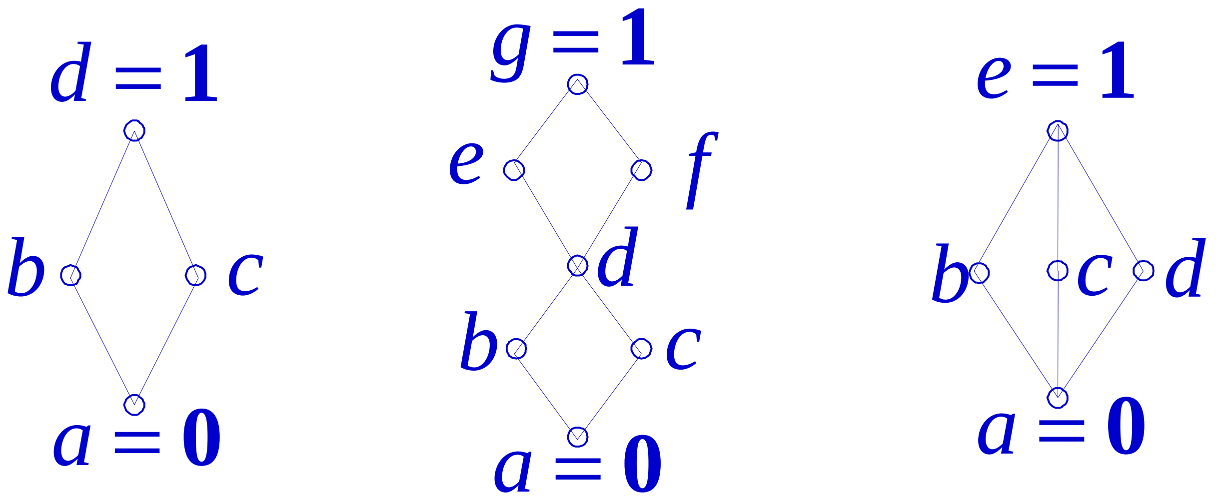
(lattice)

(Boolean algebra)

$c \cdot a = ?$

$d + b = ?$

Examples of Lattices



More Notation

- "**There Exists** a v in set V" is denoted by $\exists v \in V$

- The following are equivalent:

$a \Leftrightarrow b$ $\qquad (ab + a'b')$

$a \Rightarrow b$ **and** $a \Leftarrow b$ $\qquad ((a' + b)(a + b'))$

a **is true if and only if** b **is true**

- Does $(a' + b)(a + b')$ make sense in a poset?

No--**Complement** is defined for lattices but not for posets

Properties of Lattices

Meet, Join, Unique maximum (1), mi nimum (0) element are always defin ed

Idempotent:	$x + x = x$	$x \cdot x = x$
Commutative:	$x + y = y + x$	$x \cdot y = y \cdot x$
Associative:	$x + (y + z) = (x + y) + z$	$x \cdot (y \cdot z) = (x \cdot y) \cdot z$
Absorptive:	$x \cdot (x + y) = x$	$x + (x \cdot y) = x$

Absorptive properties are fundamental to op timization

Duality

Every lattice identity is transformed into another identity by interchanging:

- $+$ and $\cdot$
- $\leq$ and $\geq$
- $\mathbf{0}$ and $\mathbf{1}$

Example: $x \cdot (x + y) = x \;\rightarrow\; x + (x \cdot y) = x$

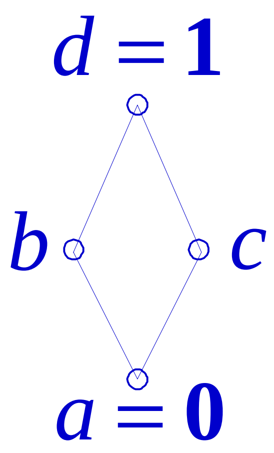
Complementation

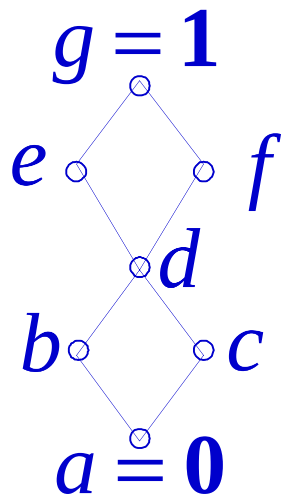
- If x+y=1 and xy=0 then x is the complement of y and vice versa
- A lattice is complemented if all elements have a complement

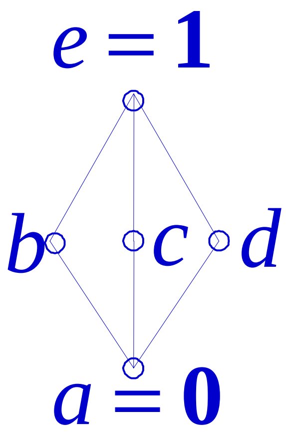
Examples of Lattices

Complemented?

yes	no	yes







$$b \cdot c = a = \mathbf{0} \qquad b \cdot c = a = \mathbf{0} \qquad b \cdot c = a = \mathbf{0}$$

$$b + c = d = \mathbf{1} \qquad b + c = d \neq \mathbf{1} \qquad b + c = e = \mathbf{1}$$

Distributivity

Semi-distributivity:

$$x \cdot (y + z) \geq (x \cdot y) + (x \cdot z)$$

$$x + (y \cdot z) \leq (x + y) \cdot (x + z)$$

Proof :

1. $x \cdot y \leq x$ (def. of meet)
2. $x \cdot y \leq y \leq y + z$ (def. of meet, join)
3. $x \cdot y \leq x \cdot (y + z)$ (def. of meet)
4. $x \cdot z \leq x \cdot (y + z)$ (mutatis mutandis: $y \leftrightarrow z$)
5. $x \cdot (y + z) \geq (x \cdot y) + (x \cdot z)$ (def. of join)

Distributivity

- **Boolean Algebras** have **full** distributivity:

$$x \cdot (y + z) = (x \cdot y) + (x \cdot z)$$

$$x + (y \cdot z) = (x + y) \cdot (x + z)$$

- **Boolean Algebras** are complemented. That is,

$$x = y' \Rightarrow (x \cdot y = \mathbf{0}) \text{ and } (x + y = \mathbf{1})$$

must hold for every x in the carrier of the **poset**

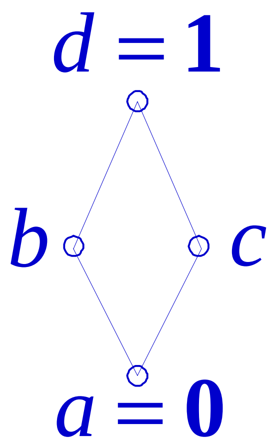
Definition of Boolean Algebra

- **A complemented, distributive lattice is a Boolean lattice or Boolean algebra**
  - **Idempotent** $x+x=x$ $xx=x$
  - **Commutative** $x+y=y+x$ $xy = yx$
  - **Associative** $x+(y+z)=(x+y)+z$ $x(yz) = (xy)z$
  - **Absorptive** $x(x+y)=x$ $x+(xy) = x$
  - **Distributive** $x+(yz)=(x+y)(x+z)$ $x(y+z) = xy +xz$
  - **Existence of the complement**

Are these Lattices Boolean Algebras?

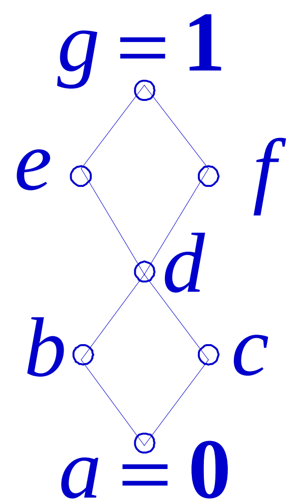
Complemented and distributed?

yes



$b \cdot c = a = \mathbf{0}$
$b + c = d = \mathbf{1}$
$a(b + c) =$
$a \cdot b + a \cdot c = a = \mathbf{0}$

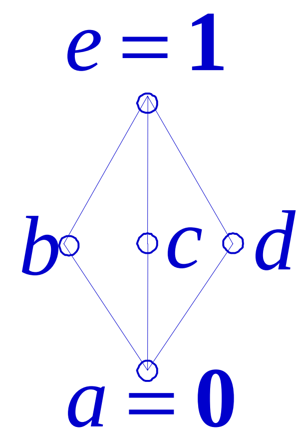
no



$b \cdot c = a = \mathbf{0}$

$b + c = d \neq \mathbf{1}$

no



$b \cdot (c + d) = b \geq$

$(b \cdot c) + (b \cdot d) = a$

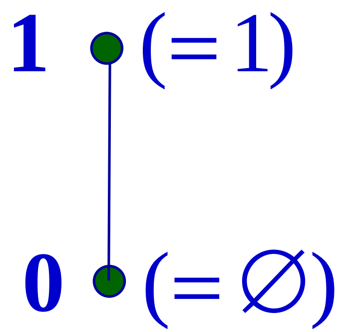
Fundamental Theorem of Boolean Algebras

- Every poset which is a Boolean Algebra has a power of 2 elements in its carrier

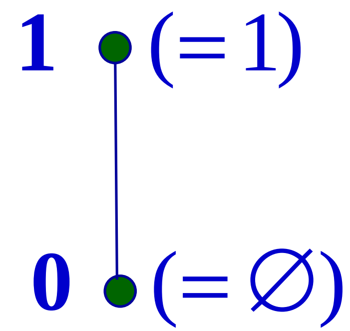
- All Boolean Algebras are isomorphic to the p ower set of the carrier.

Example:

$$V = \{1\},\ 2^{V} = \{\emptyset, 1\} = \{\mathbf{0}, \mathbf{1}\}$$



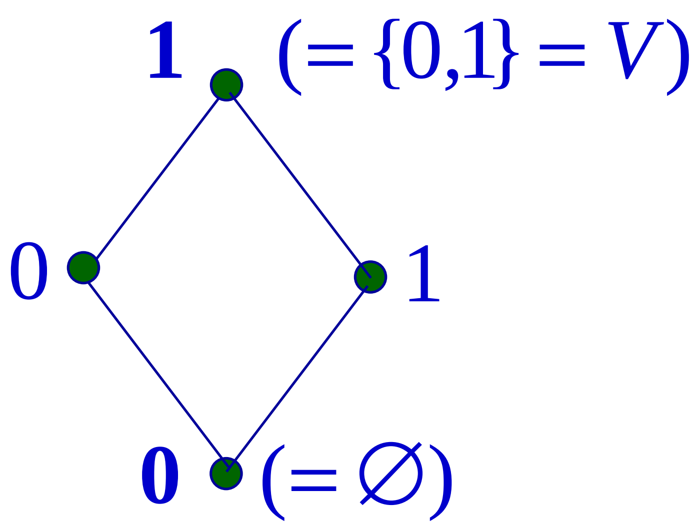
Examples of Boolean Algebras



1-cube

$$V=\{1\},\ 2^V=\{\varnothing,1\}=\{\mathbf{0},\mathbf{1}\}$$

$$V=\{0,1\},$$
$$2^V=\{\varnothing,0,1,\{0,1\}\}$$
$$=\{\mathbf{0},0,1,\mathbf{1}\}$$

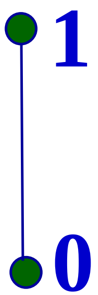


2-cube

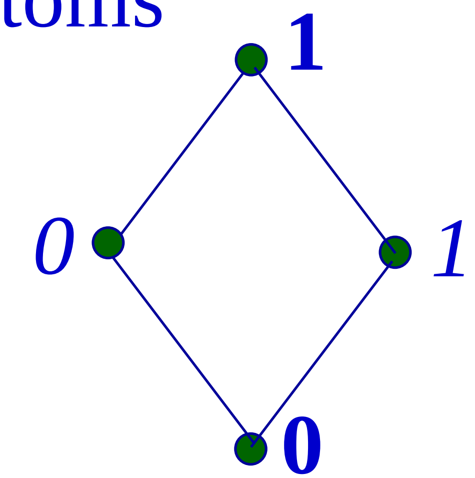
Atoms of a Boolean Algebra

- A Boolean Algebra is a Distributive, Complemented Lattice
- The minimal non-zero elements of a Boolean Algebra are called "atoms"

$$|V| = 2^n \Leftrightarrow n \text{ atoms}$$



1-atom



2-atoms

Can a Boolean Algebra have 0 atoms? ***NO!***

Examples of Boolean Algebras

$$A = \{a, b, c\}$$

$$V = 2^A$$

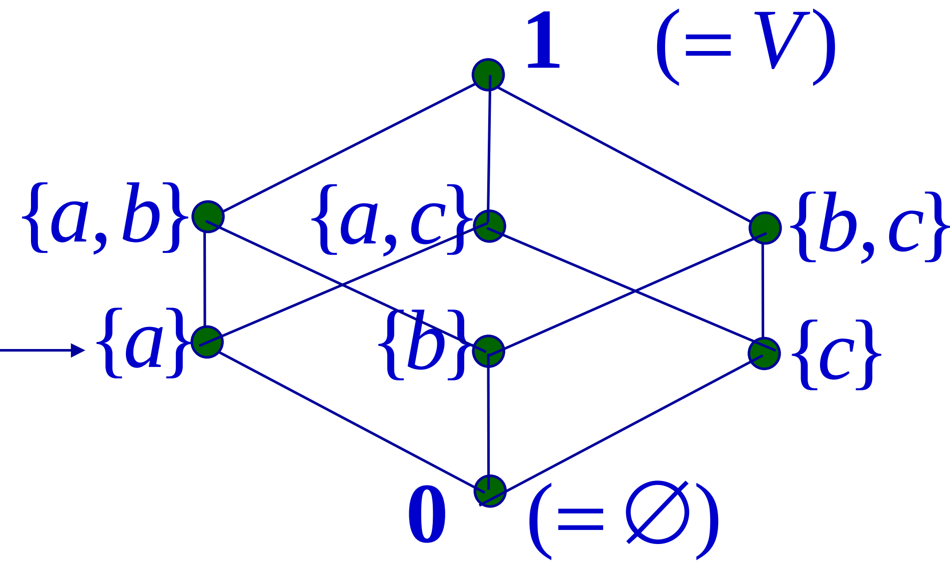
$$= \{A,$$

$$\{a,b\}, \{a,c\}, \{b,c\},$$

$$a, b, c,$$

$$\emptyset\}$$

$\longleftarrow$ 3 atoms $\longrightarrow$



Examples of Boolean Algebras

$$n = |A| = 4 \text{ atoms}, \quad n+1 = 5 \text{ levels}, \; 2^n = 16 \text{ elts}$$

$$A = \{a, b, c, d\}$$

$$V = 2^A$$

Level 4--1 elt* $= \{A,$ C_4^4

Level 3--4 elts $\{a,b,c\},\{a,b,d\},\{a,c,d\},\{b,c,d\},$ C_3^4

Level 2--6 elts $\{a,b\},\{a,c\},\{a,d\},\{b,c\},\{b,d\},\{c,d\}$ C_2^4

Level 1--4 elts $\{a\},\{b\},\{c\},\{d\}$ C_1^4

Level 0--1 elts $\varnothing\}$ C_0^4

* “elt” = element

Theorem 3.2.6 Complementation is unique.

proof: Suppose x' and y are both complements of x ($x+y=\mathbf{1},\ xy=\mathbf{0}$). Hence

$$y = y(x+x') = x'y + xy = x'y$$
$$= x'y + x'x = x'(y+x) = x'$$

Note we used distributivity. Similarly, we have

Theorem 3.2.7 (Involution): $(x')' = x$

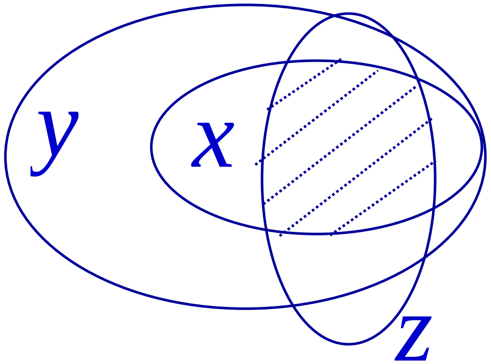
Properties of Boolean Algebra

- $x + x'y = x+y$
- $x(x'+y) = xy$

Lemma: The Isotone Property

$$x \le y \Leftrightarrow xz \le yz$$

Proof: By Theorem 3.2.2, $x \le y \Leftrightarrow x = xy$, so we get $xz = xyz = xyzz = (xz)(yz)$. Note we used idempotence, commutativity, and associativity. Then we use Theorem 3.2.2 again to prove the lemma.



$$xz \le yz \cong xz \subseteq yz$$

Duality for Boolean Algebras

Every Boolean Algebra identity is transformed into another valid identity by interchanging:

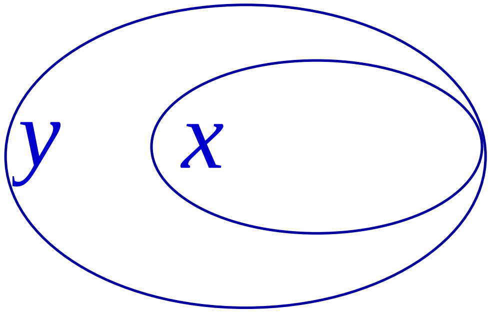
- $+$ and $\cdot$
- $\leq$ and $\geq$
- $\mathbf{0}$ and $\mathbf{1}$
- $()'$ and $()$ This rule not valid for lattices

Example: $xx' = \mathbf{0} \rightarrow x' + x = \mathbf{1}$

Theorem

$$x \le y \Leftrightarrow xy' = \mathbf{0}$$

$$\Leftrightarrow x' + y = \mathbf{1}$$



Proof: **By the isotone property we have**

$$x \le y \Leftrightarrow xy' \le yy' \Leftrightarrow xy' \le \mathbf{0} \Leftrightarrow xy' = \mathbf{0}$$

The second identity follows by duality.

Other Theorems

DeMorgan's Laws

$$(x+y)' = x'y'$$

$$(xy)' = x'+y'$$

Consensus

$$xy + x'z + yz = xy + x'z$$

$$(x+y)(x'+z)(y+z) = (x+y)(x'+z)$$

Consensus Examples

1. $a'bc + abd + bcd = abc + a'bd$

3. $abe + bce + bde + ac'd'$

$= abe + be(c + d) + a(c + d)'$

$= be(c + d) + a(c + d)'$

$= bce + bde + ac'd'$

Note use of DeMorgan's Law

Consensus Example

5. Is wrong. Replace by

$$a'c'd + b'c'd + acd + bcd$$

$$= (a'bd) + a'c'd + b'c'd + acd + bcd$$

$$= a'bd + b'c'd + acd$$

- **Uphill Move: Note addition of redundant consensus term enables deletion of two other terms by consensus**
- **This avoids local minima--a crucial part of the logic minimization paradigm**

Functions and Boolean Functions

Ordinary **functions** of 1 variable:

$$f(x): D \mapsto R \Leftrightarrow f \subseteq D \times R$$

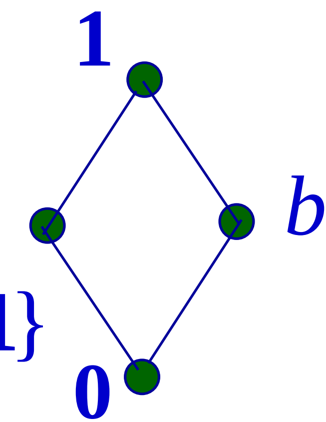
Ordinary **functions** of 2 real variables:

$$f(x,y): D_x \times D_y \mapsto R \Leftrightarrow f \subseteq (D_x \times D_y) \times R$$

Boolean functions of n variables:

$$f(x_1,\dots,x_n): B^n \mapsto B, \qquad B = \{\mathbf{0},\dots,\mathbf{1}\}$$

$$f(x_1,\dots,x_n) \subseteq (B \times \cdots \times B) \times B$$



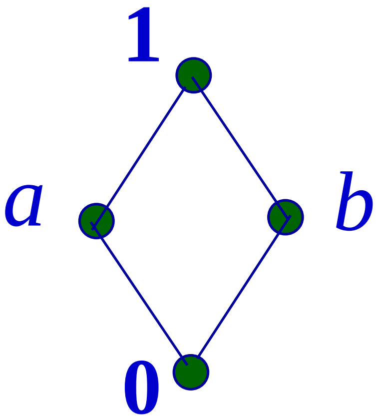
Boolean Formulae: Meets and Joins of Variables and Constants

$$F_1 = 0$$

$$F_2 = x_1 x_2' + x_1' x_2$$

$$F_3 = x_1 x_1 x_2' + x_1' x_2$$

$$F_3 = a x_1 + b ???$$



n-variable Boolean Formulae

$$B = \{\mathbf{0}, \ldots, \mathbf{1}\}$$

- The elements of B are Boolean formulae.
- Each variable $x_1, \ldots, x_n$ is a Boolean formula.
- If g and h are Boolean formulae, then so are
  - $g + h$
  - $g \cdot h$
  - g'
- A string is a Boolean formula if and only if it derives from finitely many applications of these rules.

Distinct Formulas, Same Function

$$B = \{\mathbf{0}, a, b, \mathbf{1}\}$$

Truth Table for $f : B^2 \mapsto B$

x_1	0000	aaaa	bbbb	1111
x_2	0ab1	0ab1	0ab1	0ab1
$F = x_1 + x_2$	0ab1	aa11	b1b1	1111
$G = x_1 + x_1' x_2$	0ab1	aa11	b1b1	1111

Boolean Functions on Boolean Algebra

$$B = \{\mathbf{0}, \ldots, \mathbf{1}\}$$

- $f(x_1, \ldots, x_n) = x_i$ is a Boolean function
- $f(x_1, \ldots, x_n) = e \in B$ is also
- If g and h are Boolean, functions then so are
  - $g + h$
  - $g \cdot h$
  - g'
- A function is Boolean if and only if it derives from finitely many applications of these rules.

Cofactors

Positive Cofactor WRT x_1: $f_{x_1} = f(\mathbf{1}, x_2, \ldots, x_n)$

Negative Cofactor WRT x_1: $f_{x_1'} = f(\mathbf{0}, x_2, \ldots, x_n)$

Note: prime denotes complement

$$f_{x_1} = f(\mathbf{1}, x_2, \ldots, x_n) = f_{x_1 = \mathbf{1}}$$

$$f_{x_1'} = f(\mathbf{0}, x_2, \ldots, x_n) = f_{x_1 = \mathbf{0}}$$

Cofactors

Positive Cofactor WRT x_1: $f_{x_1} = f(\mathbf{1}, x_2, \ldots, x_n)$

Negative Cofactor WRT x_1: $f_{x_1'} = f(\mathbf{0}, x_2, \ldots, x_n)$

Example:

$$f = abc'd + a'cd' + bc$$

$$f_a = bc'd + \qquad bc$$

$$f_{a'} = \qquad + cd' + bc$$

This term drops out when a is replaced by **1**

This term is unaffected

Cofactors

$$B = \{\mathbf{0}, a, b, \mathbf{1}\}, \quad f(x): B^n \mapsto B$$

$$f_{x_1=a} = f(a, x_2, \ldots, x_n)$$

$$f_{x_1} = f(\mathbf{1}, x_2, \ldots, x_n)$$

Example:

$$f = ax_1' + bx_2$$

$$f_{x_1=a} = aa' + bx_2 = \mathbf{0} + bx_2 = bx_2$$

Boole's Expansion Theorem

$$f(x_1, x_2, \ldots, x_n) =$$

$$x_i f_{x_i} + x_i' f_{x_i'} = [x_i + f_{x_i'}] \cdot [x_i' + f_{x_i}]$$

Example: $f = a x_1' + b x_2,$

$$f_{x_1} = a\mathbf{0} + b x_2 = b x_2, \quad f_{x_1'} = a\mathbf{1} + b x_2 = a + b x_2$$

Sum form: $f = x_1(b x_2) + x_1'(a + b x_2)$

Product form: $f = [x_1 + (a + b x_2)] \cdot [x_1' + (b x_2)]$

Variables and Constants

- In the previous slide a and b were constants, and the $x_1,\ldots,x_n$ were variables.
- But we can also use letters like a and b as variables, without explicitly stating what the elements of the Bo olean Algebra are.

Boole's Expansion (Sum Form)

$$f = abc'd + a'cd' + bc$$

$$f_a = bc'd + bc$$

$$f_{a'} = cd' + bc$$

$$f = af_a + a'f_{a'} = a(bc'd + bc) + a'(cd' + bc)$$

Boole's Expansion (Product Form)

$$f = abc'd + a'cd' + bc$$

$$f = [a + f_{a'}][a' + f_a]$$

$$= [a + (cd' + bc)][a' + (bc'd + bc)]$$

$$= aa' + a(bc'd + bc) + a'(cd' + bc) +$$

$$(cd' + bc)(bc'd + bc)$$

$$= abc'd + a'cd' + bcd' + bc$$

Note application of absorptive law

$$= abc'd + a'cd' + bc$$

Minterm Canonical Form

The minterm canonical form is a canonical, or standard way of representing functions. From p100, $f = x + y' + z$ is represented by millions of distinct Boolean formulas, but just 1 minterm canonical form. Note

$$f = x + y' + z = (x'yz')'$$

Thus some texts refer to f as {0,1,3,4,5,6,7}

$$f = x'y'z' + x'y'z + x'yz + xy'z + \cdots$$
$$(0, \quad 1, \quad 3, \quad 4, \cdots)$$

Minterm Canonical Form

n levels of recursive cofactoring create 2^n constants

$$f(x_1, x_2, \ldots, x_n)$$

$$= x_1' f_{x_1'} + x_1 f_{x_1}$$

$$x_1'x_2' f_{x_1'x_2'} + x_1'x_2 f_{x_1'x_2} + x_1x_2' f_{x_1x_2'} + x_1x_2 f_{x_1x_2}$$

$$\ddots$$

$$= x_1' \cdots x_n' f_{x_1' \cdots x_n'} + x_1' \cdots x_{n-1}' x_n f_{x_1' \cdots x_{n-1}' x_n} + \cdots$$

$$+ x_1' x_2 \cdots x_n f_{x_1' x_2 \cdots x_n} + x_1 \cdots x_n f_{x_1 \cdots x_n}$$

These elementary functions are called minterms

Minterm Canonical Form

Thus a Boolean function is uniquely determined by its values at the corner points $0\cdots0, 0\cdots01, \ldots, 1\cdots1$

$$
\begin{aligned}
& f(x_1, x_2, \ldots, x_n) \\
&= x_1' f_{x_1'} + x_1 f_{x_1} \\
& \quad x_1'x_2' f_{x_1'x_2'} + x_1'x_2 f_{x_1'x_2} + x_1x_2' f_{x_1x_2'} + x_1x_2 f_{x_1x_2} \\
& \quad \ddots \\
&= x_1'\cdots x_n' f_{x_1'\cdots x_n'} + x_1'\cdots x_{n-1}'x_n f_{x_1'\cdots x_{n-1}'x_n} + \cdots \\
& \quad + x_1'x_2\cdots x_n f_{x_1'x_2\cdots x_n} + x_1\cdots x_n f_{x_1\cdots x_n}
\end{aligned}
$$

These 2^n constants are aptly called **discrimants**

Minterm Canonical Form

$$f : B^2 \mapsto B \quad B = \{\mathbf{0}, a, b, \mathbf{1}\}$$

$$f = ax_1' + bx_2, \quad f_{x_1'} = a + bx_2, \quad f_{x_1} = bx_2$$

$$= x_1'x_2'a + x_1'x_2(a+b) + x_1x_2'\mathbf{0} + x_1x_2b$$

This function is Boolean (from Boolean Formula)

x_1	0000	aaaa	bbbb	1111
x_2	0ab1	0ab1	0ab1	0ab1
$F = ax_1' + bx_2$	aa11	00bb	aa11	00bb
$x_1'x_2'a + x_1'x_2 + x_1x_2b$	aa11	00bb	aa11	00bb
	⇑ ⇑			⇑ ⇑

Thus all 16 cofactors match--**not just discriminants**

Now Suppose Truth Table is Given

- Here we change 15th cofactor, but leave the 4 discriminants unchanged
- Since the given function doesn't match at all 16 cofactors, *F* is **not Boolean**

x_1	0000	aaaa	bbbb	1111
x_2	0ab1	0ab1	0ab1	0ab1
F	aa11	00bb	aa11	001b
$x_1'x_2'a + x_1'x_2 + x_1x_2b$	aa11	00bb	aa11	00bb
	⇑ ⇑			⇑ ⇑

2-Variable Boolean Functions

Minterms	2-variable functions
4	$f^{15} = \mathbf{1}$
3	$f^{11\text{–}14} = x' + y', x' + y, x + y', x + y$
2	$f^{5\text{–}10} = x', y', x, y, xy' + x'y, xy + x'y'$
1	$f^{1\text{–}4} = x'y', xy', x'y, xy$
0	$f^{0} = \mathbf{0}$

Note that the minterms just depend on the number and names of the variables, independent of the particular Boolean Algebra

Boolean Function Algebras

The notation Fn(B) means "the Boolean Algebra whose ca rrier is the set of all n-variable Boolean Functions" which map B into Fn(B): Bn |-> B minterms $F_n(B): B^n \mapsto B$

If $B = \{0,1\}$ **the atoms of Fn(B) are its n-variable**

B is called the "<u>Base Algebra</u>" of the Boolean Function al gebra

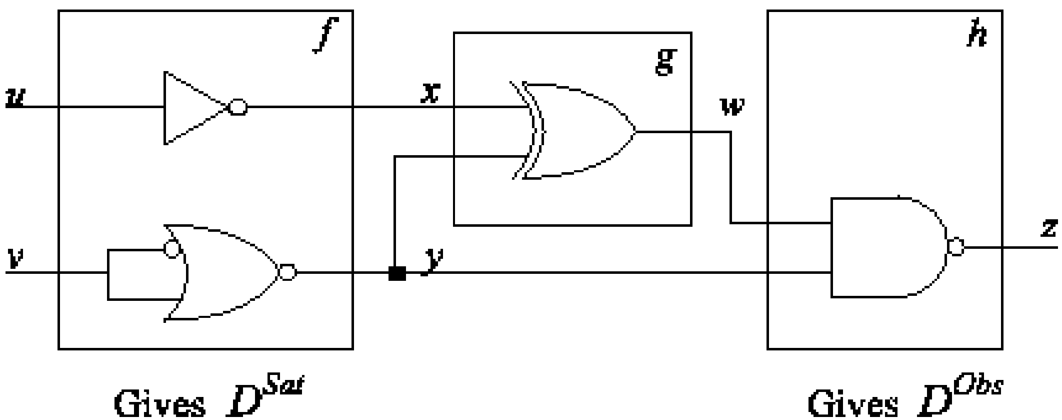
The atoms of B are called <u>Base Atoms</u>

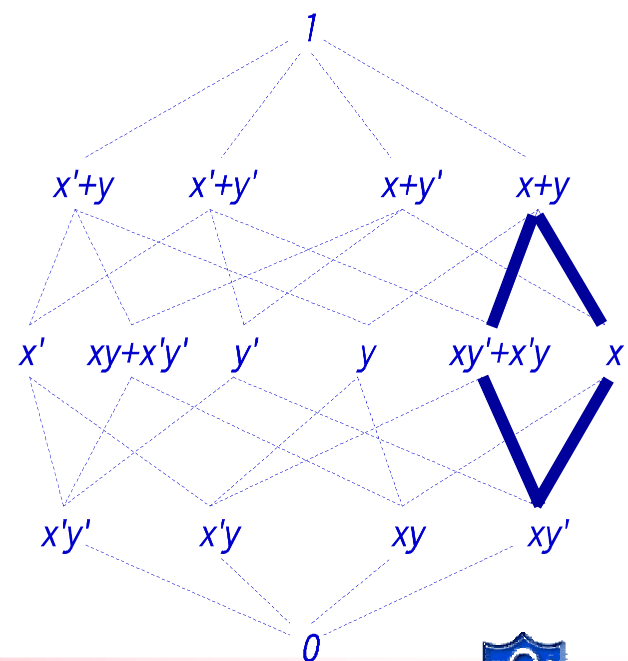
Large Boolean Algebras?

• When you design an optimal circuit, each gate must be optimized with respect to its Don't Cares

• Because of Don't Cares, 4 functions of (x, y) are equivalence preserving replacements for gate g

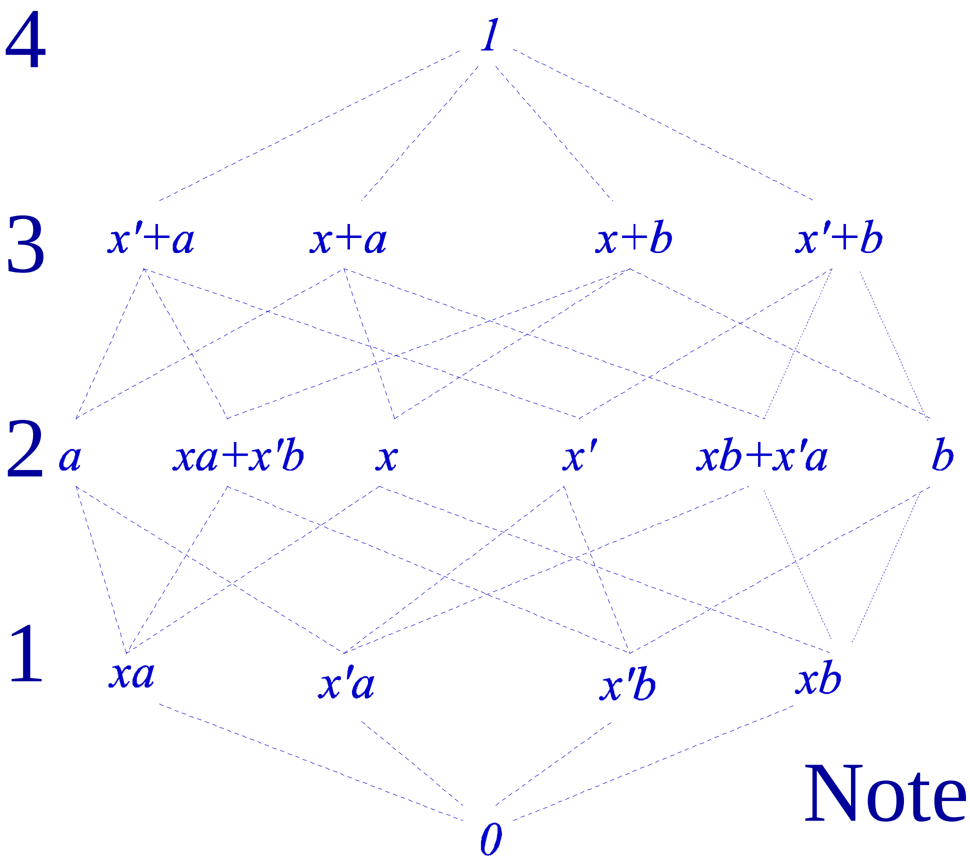
• Optimal Design: pick best such replacement





$$F_1(\{\mathbf{0}, a, b, \mathbf{1}\})$$

Level (= # of atoms)



atoms of $F_1\{\mathbf{0}, a, b, \mathbf{1}\})$ are base atoms $\cdot$ minterms

Each element has 3 Atoms

Each element has 2 Atoms

Each element has 1 Atom

Note: base atoms act like 2 extra literals $a \approx y,\ a' = b \approx y'$

Boolean Algebra

$F_2(\{\mathbf{0},\mathbf{1}\})$

poset: $\{f^0, \ldots, f^{15}\}$

$f^i \le f^j \Leftrightarrow$

$\text{minterms}(f^i) \subseteq \text{minterms}(f^j)$

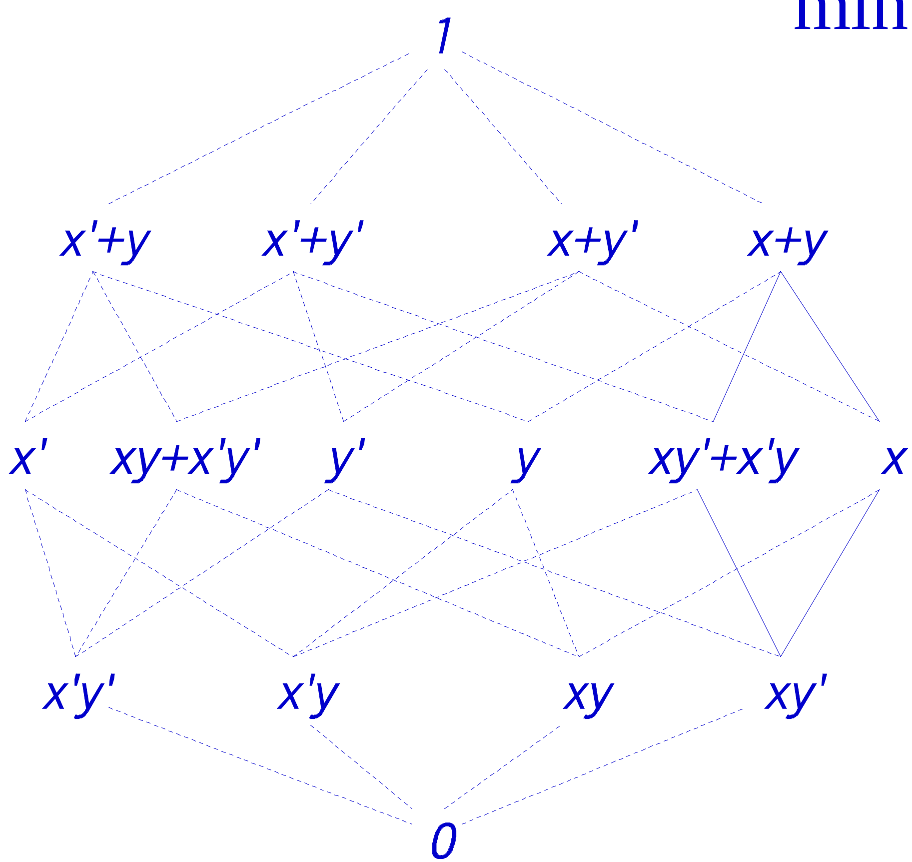
Level (= # of atoms)

4

3

2

1

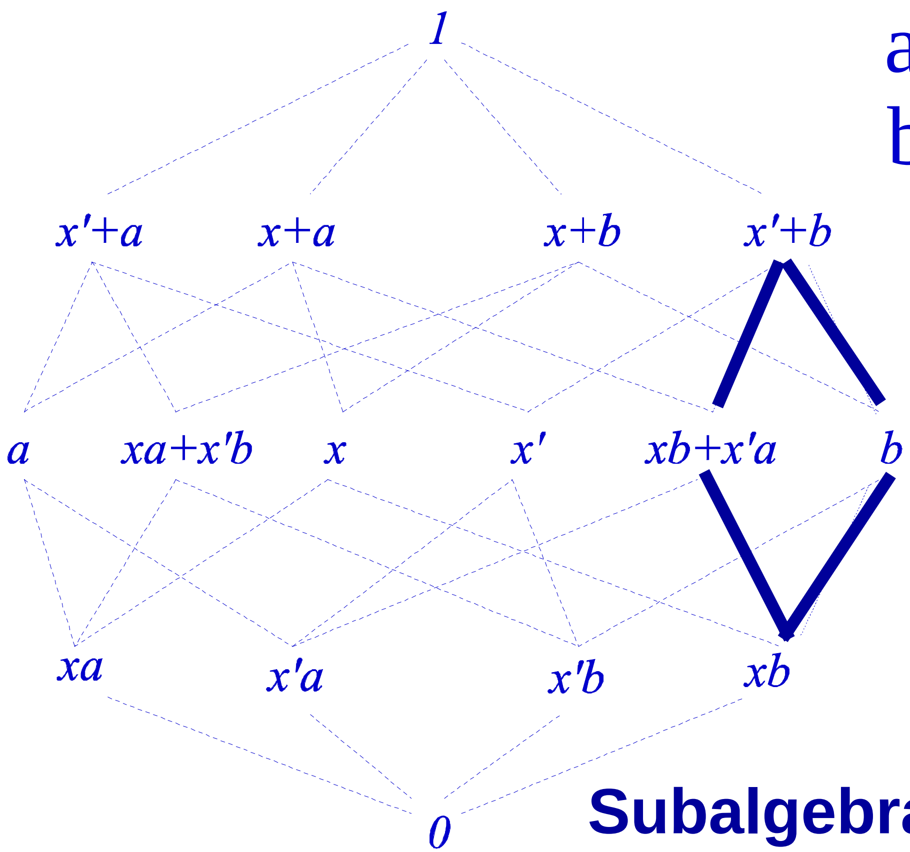


3 minterms (3 atoms)

2 minterms (2 atoms)

1 minterm (1 atom)

Boolean Algebra $F_1(\{\mathbf{0},a,b,\mathbf{1}\})$ of 1-Variable Functions



atoms of $F_1\{\mathbf{0},a,b,\mathbf{1}\})$ are base atoms $\cdot$ minterms

3 Atoms

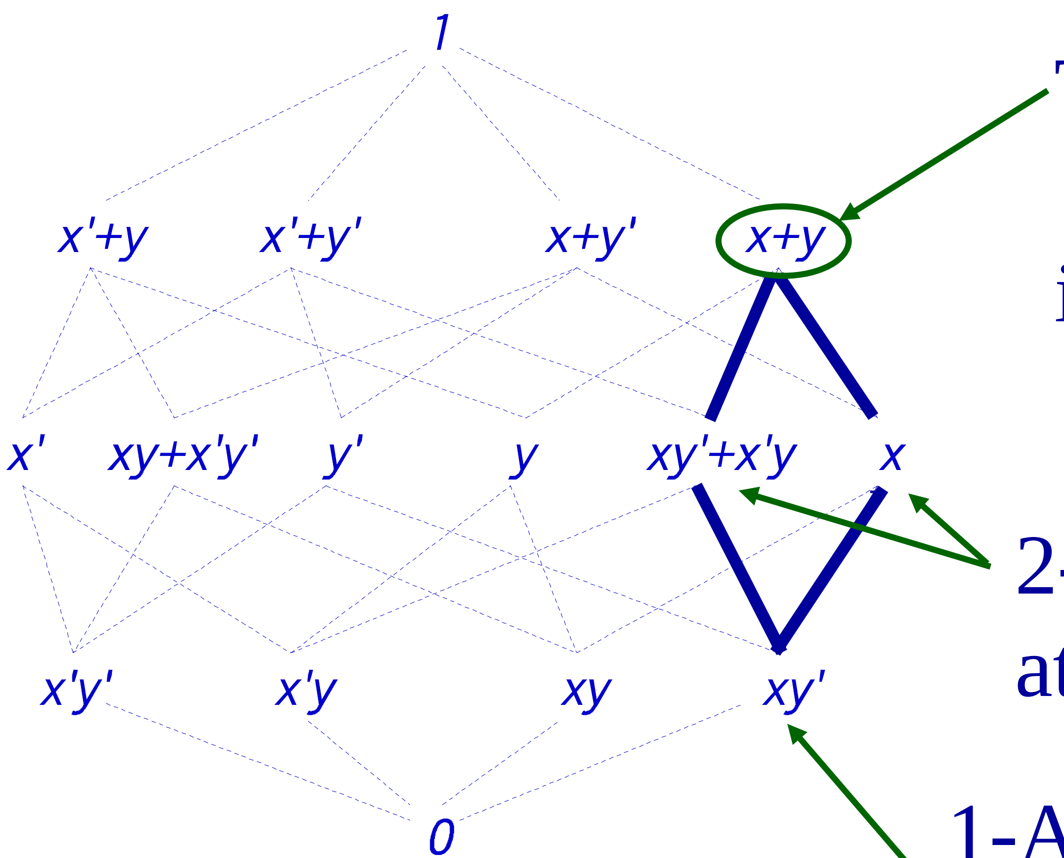
2 Atoms

1 Atom

Subalgebras are <u>incompletely specified Boolean Functions</u> (most important)

Boolean Subalgebra

Boolean Subalgebra: Interval of 2-Variable Function Lattice

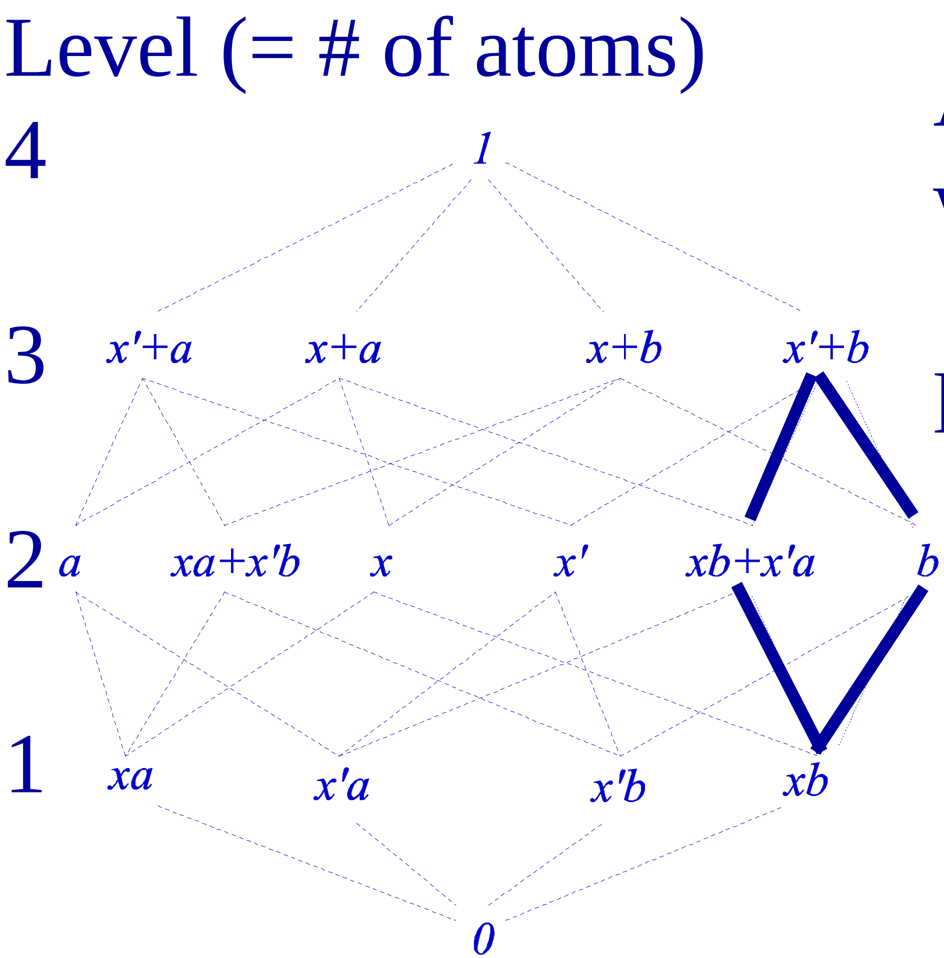


The 3-Atom element $x + y = xy' + xy + x'y$ is **ONE** of subalgebra

2-Atom elements are atoms of subalgebra

1-Atom element xy' is **ZERO** of subalgebra

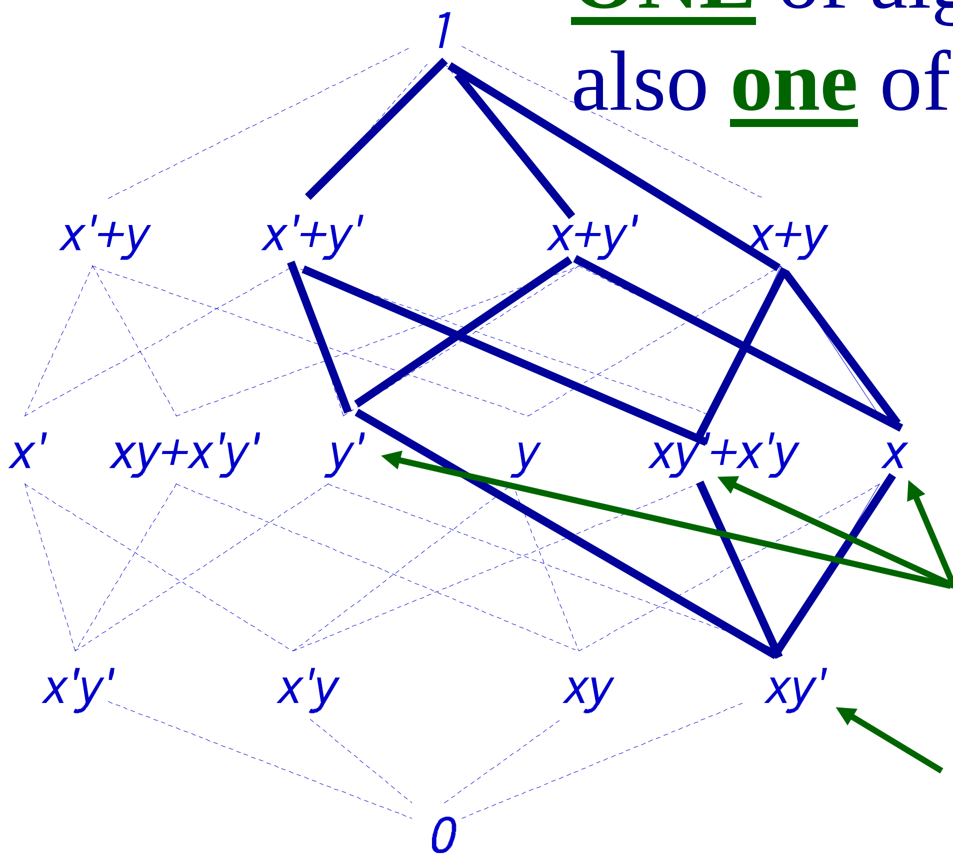
More On Counting



An algebra (or subalgebra) with $n+1$ levels has exactly 2^n elements, because $n+1$ levels implies n atoms

A larger Subalgebra

Interval of 2-Variable Function Lattice



ONE of algebra (4-Atoms) is also **one** of this subalgebra

2-Atom elements are atoms of subalgebra

1-Atom element is **ZERO** of subalgebra

Counting Elements of $F_n(B)$

- For each of the 2^n minterms, the discriminant can be chosen as any of the $|B|$ elements of B.
- Therefore, the number of elements of $F_n(B)$ is
- Examples

$$|B|^{(2^n)} = (2^{|A(B)|})^{(2^n)} = 2^{(|A(B)| \cdot 2^n)}$$

$$B = \{\mathbf{0},\mathbf{1}\},\ |A(B)| = 1$$

$$n = 2 \Rightarrow |B|^{(2^n)} = 2^{2^2} = 2^4 = 16$$

$$n = 3 \Rightarrow |B|^{(2^n)} = 2^{2^3} = 2^8 = 256$$

Counting Elements of Fn(B)

- **Examples with** $B = \{\mathbf{0}, a, b, \mathbf{1}\},\ A(B) = \{a, b\}$

$$n = 2 \Rightarrow 2^{(|A(B)| \cdot 2^n)} = 2^{2 \cdot 2^3} = 2^{16}$$

$$n = 3 \Rightarrow 2^{(|A(B)| \cdot 2^n)} = 2^{2 \cdot 2^4} = 2^{32}$$

- **Examples with**

$$|A(B)| = 4$$

- **Note 2(4) base atoms act like 1(2) extra variables**

$$n = 2 \Rightarrow 2^{(|A(B)| \cdot 2^n)} = 2^{4 \cdot 2^3} = 2^{32}$$

$$n = 3 \Rightarrow 2^{(|A(B)| \cdot 2^n)} = 2^{4 \cdot 2^4} = 2^{64}$$

Boolean Difference (Sensitivity)

A Boolean function f depends on a if and only if $f_a \neq f_{a'}$. Thus

$$\partial f / \partial a = f_a \oplus f_{a'}$$

is called the Boolean Difference, or Sensitivity of f with respect to a

Boolean Difference (Sensitivity)

Example:

$$f = abc + a'bc$$

$$\partial f / \partial a = f_a \oplus f_{a'} = (bc) \oplus (bc) = \mathbf{0}$$

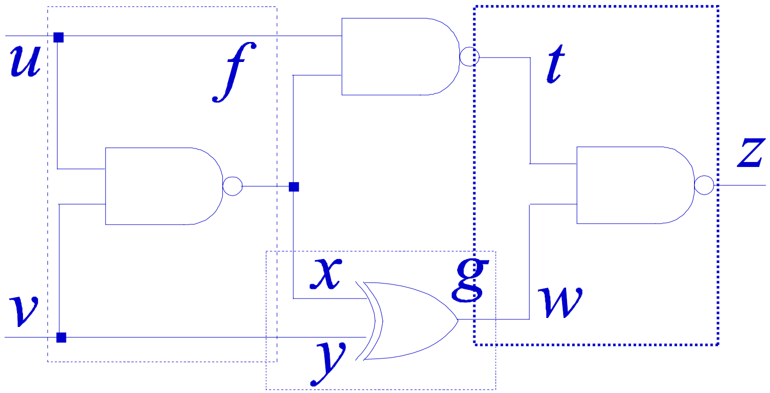
$$\partial f / \partial b = f_b \oplus f_{b'} = (ac) \oplus (a'c) = c$$

Note the formula depends on a, but the implied function does not

Don’t Care

- **An interval [L, U] in a Boolean algebra B is the subset of B defined by [L,U] = { x $\in$ B : L $\leq$ x $\leq$ U }**
- **Satisfiability don’t cares**
- **Observability don’t cares**

Intervals and Don't Cares



$$D = D^{Obs} + D^{Sat} = x'y'$$

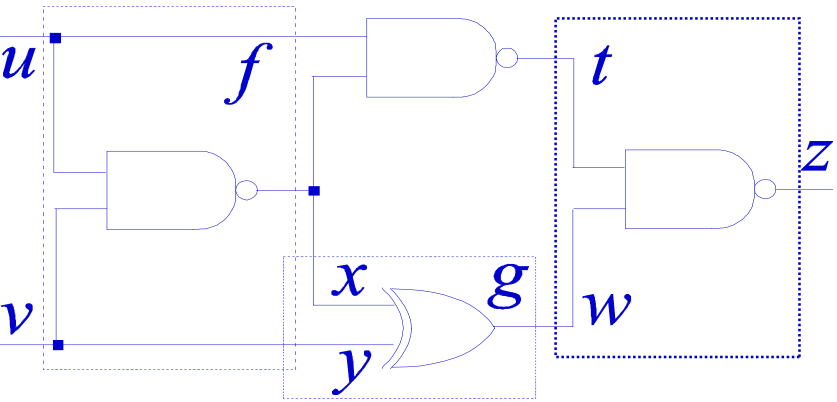
$$L = g - D = gD'$$

$$= (x'y + xy')(x + y) = (x'y + xy') = g$$

$$U = g + D = (x'y + xy') + x'y'$$

$$= x' + y'$$

Don't Cares



Input Filter

Output Filter

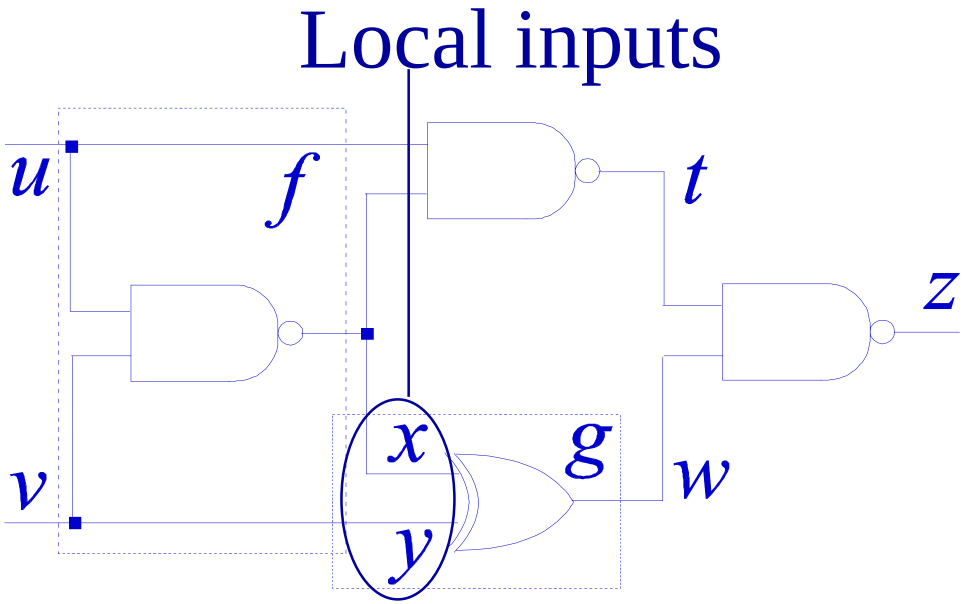
$xy \in D^{Obs}$ if and only if $f^x = x,\ f^y = y \Rightarrow \partial z / \partial w = \mathbf{0}$ for all possible xy.

$xy \in D^{Sat}$ if and only if local input xy never occurs

The complete don't care set for gate g is

$$D^g = D^{Sat} + D^{Obs}$$

Don’t Cares



$$x = u' + v', \quad y = v$$

$$(y = \mathbf{0}) \Rightarrow (x = \mathbf{1})$$

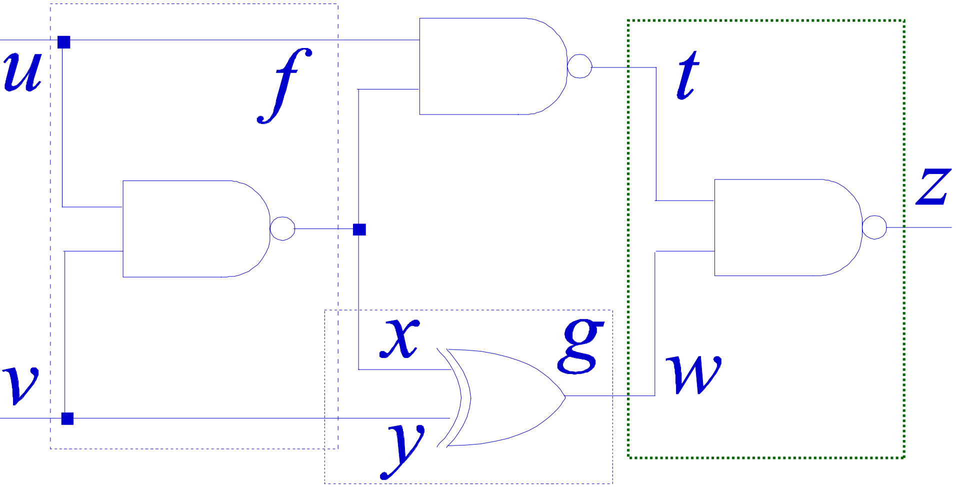
For this circuit, local input combinations

$x'y' \ (x = 0, \ y = 0)$

do not occur. That is, the local minterm

$x'y'$ is don’t care.

Don’t Cares



$$z = w' + t'$$

$$\partial z / \partial w = z_w \oplus z_{w'}$$

$$= t' \oplus \mathbf{1}$$

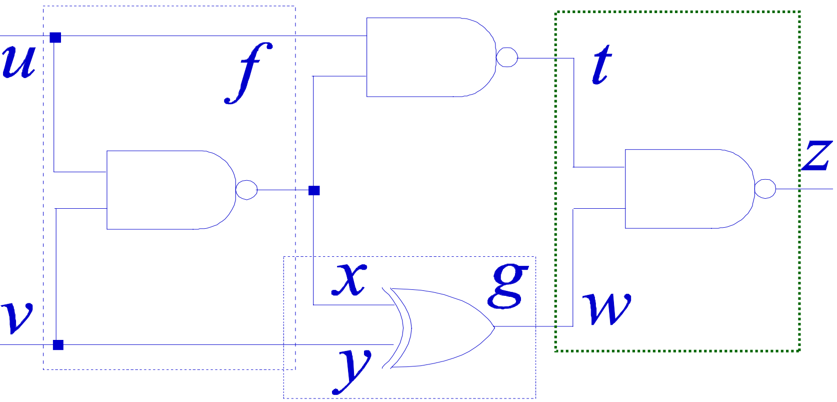
$$= t$$

For this circuit, global input combination 10 sets

$$t' = uv' = \mathbf{1}$$

which makes z insensitive to w. However, local input pair 10 (xy') is **NOT** don’t care, since $u'v'$ also gives xy', and in this case $t_{u'v'} = 1$.

Computing ALL Don't Cares

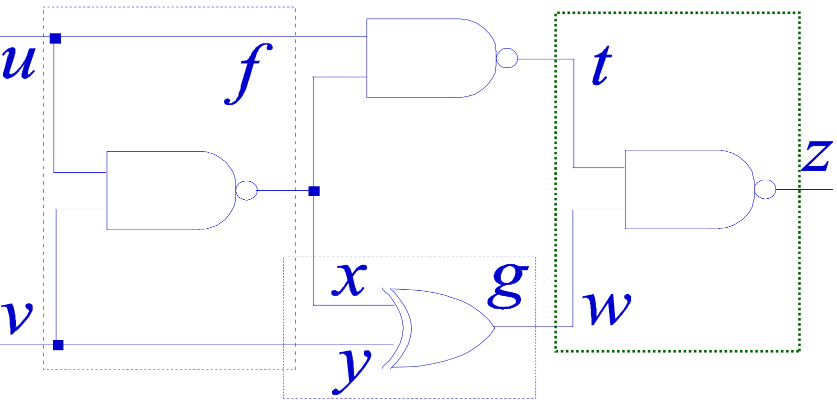


u	v	x	y	w	t	z	$\partial z / \partial w$
0	0	1	0	1	1	0	1
0	1	1	1	0	1	1	1
1	0	1	0	1	0	1	0
1	1	0	1	1	1	0	1

$xy \in D^{Sat}$ if and only if $f^x(u,v) = x$ and $f^y(u,v) = y$ does not occur for any row u,v in the truth table.

Here, $D^{Sat} = x'y'$ (**00** does not occur)

Computing ALL Don't Cares



Note these differ!

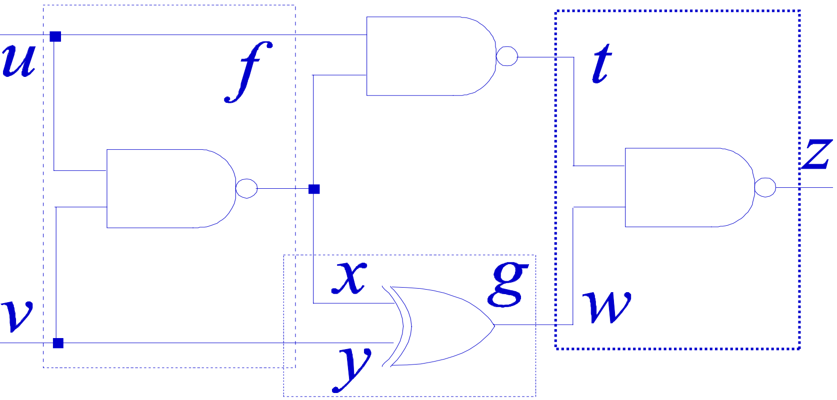
u	v	x	y	w	t	z	$\partial z/\partial w$
0	0	1	0	1	1	0	1
0	1	1	1	0	1	1	1
1	0	1	0	1	0	1	0
1	1	0	1	1	1	0	1

Similarly $xy \in D^{Obs}$ if and only for every row u,v such that $f^x(u,v)=x$ and $f^y(u,v)=y$,

$$\partial z/\partial w = z_w \oplus z_{w'} = \mathbf{0}$$

Here 10 (xy') is **NOT** don't care since $\partial z/\partial w = \mathbf{1}$ in the first row.

Computing ALL Don't Cares



Note these differ!

u	v	x	y	w	t	z	$\partial z / \partial w$
0	0	1	0	1	1	0	1
0	1	1	1	0	1	1	1
1	0	1	0	1	0	1	0
1	1	0	1	1	1	0	1

$$D^g = D^{Sat} + D^{Obs} = x'y'$$

$$g = xy' + x'y \ (+x'y') \rightarrow x' + y'$$

Thus the exclusive OR gate can be replaced by a NAND

Don’t Cares and Function Subalgebras

Suppose we are given a Boolean Function f and a don’t care set D. Then any function in the interval (subalgebra)

$$[f_L, f_U] = [fD', f + D]$$

is an acceptable replacement for f in the environment that produced D. Here fD' is the **0** of the subalgebra and $f + D$ is the **1**.

Incompletely Specified Functions

Suppose we are given a Boolean Function g and a don't care set D. Then the triple

$$(f, d, r)$$

where $f = gD'$, $d = D$, and $r = (f + D)'$ is called an **Incompletely specified function.**

Note $f + d + r = gD' + D + (g + D)' = \mathbf{1}$.